Optional Mathematics

Grade – 9

Government of Nepal

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Preface

The curriculum and curricular materials have been developed and revised on a regular basis with the aim of making education objective-oriented, practical, relevant and job oriented. It is necessary to instill the feelings of nationalism, national integrity and democratic spirit in students and equip them with morality, discipline and self-reliance, creativity and thoughtfulness. It is essential to develop in them the linguistic and mathematical skills, knowledge of science, information and communication technology, environment, health and population and life skills. it is also necessary to bring in them the feeling of preserving and promoting arts and aesthetics, humanistic norms, values and ideals. It has become the need of the present time to make them aware of respect for ethnicity, gender, disabilities, languages, religions, cultures, regional diversity, human rights and social values so as to make them capable of playing the role of responsible citizens. This textbook for grade nine students as an optional mathematics has been developed in line with the Secondary Level Optional Mathematics Curriculum, 2074 so as to strengthen mathematical knowledge, skill and thinking on the students. It is finalized by incorporating recommendations and feedback obtained through workshops, seminars and interaction programmes.

The textbook is written by Mr. Hari Narayan Upadhyaya, Mr. Nara Hari Acharya and Mr. Med Nath Sapkota. In Bringing out the textbook in this form, the contribution of the Director General of CDC Dr. Lekha Nath Poudel is highly acknowledged. Similarly, the contribution of Prof. Dr. Ram Man Shrestha, Mr. Laxmi Narayan Yadav, Mr. Baikuntha Prasad Khanal, Mr. Krishna Prasad Pokharel, Mr. Anirudra Prasad Neupane, Ms. Goma Shrestha, Mr. Rajkumar Mathema is also remarkable. The subject matter of the book was edited by Dr. Dipendra Gurung and Mr. Jagannath Adhikari. The language of the book was edited by Mr. Nim Prakash Singh Rathaur. The layout of this book was designed by Mr. Jayaram Kuikel. CDC extends sincere thanks to all those who have contributed to developing this textbook.

This book contains various mathematical concepts and exercises which will help the learners to achieve the competency and learning outcomes set in the curriculum. Efforts have been made to make this textbook as activity-oriented, interesting and learner centered as possible. The teachers, students and all other stakeholders are expected to make constructive comments and suggestions to make it a more useful textbook.

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Table of Content

Unit 1

Algebra

1.0 Review

Provide two dice with color blue and red to students. Ask the students to roll two times successively and tabulate the result taking numbers in the blue die in the first element (x) and the number in the red die as second element (y). For example,

x	3	5
y	4	6

Plot theses points on the graph, draw a line through them and complete the following tasks.

1. Write an equation to represent the above line.
2. Does the point (6, 6) lie on this line? If it does not lie on the line, write an inequality that is satisfied by the point (6, 6).
3. If a point satisfies the equation, identify it as a solution of the equation.
4. Repeat this activity at least four times for different points.

1.1 Relation and Functions

a) Ordered Pairs:

In day to day life there are situations where position of the objects matters. For example, consider the two numbers 5 and 7. Taking 5 as the first and 7 as the second and working out their difference.

5 – 7 = – 2 (negative two)

Taking 7 as the first and 5 as the second and working out their difference, we get.

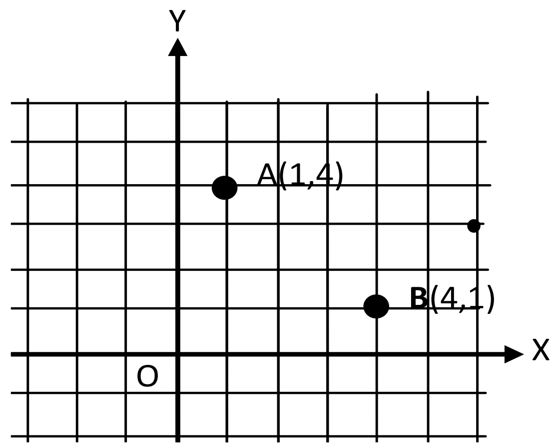
7 – 5 = 2 (Positive two)

Here, the number –2 and 2 are different. Hence, order of number matters in most cases.

The pair of numbers a and b; where a is always in the first position and b is always in the second position is called the ordered pair of numbers a and b. It is denoted by (a, b).

Example 1

Graph the ordered pairs A(1, 4) and B(4, 1). Based on the position of A(1, 4) and B(4, 1) in the graph, discuss on their difference.



Solution:

The ordered pairs A(1, 4) and B(4, 1) when graphed, they represent different positions. Hence the ordered pairs are not equal. i. e, (1, 4) ≠ (4, 1).

In coordinates system, in the ordered pairs (a, b), we take the first component 'a' as the x–coordinate and the second component 'b' as the y – coordinate.

b) Equality of two ordered pairs:

Two ordered pairs (a, b) = (c, d) if and only if (iff) a = c and b = d.

Two ordered pairs are equal if and only if their corresponding components are equal.

Example 2

Find the values of x and y if

a) (x, 7) = (– 2, y)

b) (x–2y, 9) = (2, 2x + y)

Solution:

a) Equating the corresponding components of the equal ordered pair of numbers

(*x*, 7) = (–2, y), we get

x = –2

And y = 7.

b) Equating the corresponding components of the equal ordered pair of numbers.

(*x*–2y, 9) = (2, 2*x* + y) we get

x –2y = 2………. (i)

$2x + y = 9$………….. (ii)

From (i) $x = 2y + 2$……… (iii)

Substituting the value of x from equation (iii) in equation (ii) we get

$2(2y + 2) + y = 9$

Or, $4y + 4 + y = 9$

Or, $5y = 5$

Or, $y = 1$.

Now from equation (iii), putting $y = 1$, we get

$x = 2 \times 1 + 2$

Or, $x = 4$.

Hence, $x = 4$, and $y = 1$ are the required solutions.

Exercise: 1.1

1. (a) Define "ordered pair".

(b) Give an example of ordered pair of numbers.

(c) When are the two ordered pairs equal?

(d) If (a, b) = (2, 4) what are the values of a and b?

2. Which of the following order pairs are equal?

(a) (3, 4) and (4, 3)

(b) $(2 - 1, 5 + 1)$ and $(5 - 4, \frac{12}{2})$

(c) $(18 \div 3, 4 \times 2)$ and $(2 \times 3, 5 + 2)$

(d) $(4 + 5, 21 \div 7)$ and $(3 \times 3, 4 - 1)$

3. Find the values of x and y in each of the following

(a) $(x, 4) = (5, y)$

(b) $(x - 1, y + 2) = (6, 7)$

(c) $(x - 3, y + 7) = (2, 5)$

(d) $(2x - 5, 4) = (9, y + 4)$

(e) $\left(\frac{x}{3} + 1, y - \frac{2}{3}\right) = \left(\frac{5}{3}, \frac{1}{3}\right)$

4. **Find the values of x and y in each of the following.**

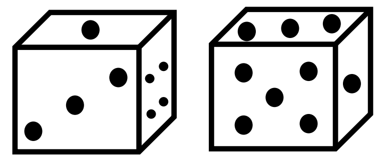
(a) $(3x + 5y, 17) = (11, 6x + 5y)$

(b) $(3x + 2y, 1) = (5, 2x - y)$

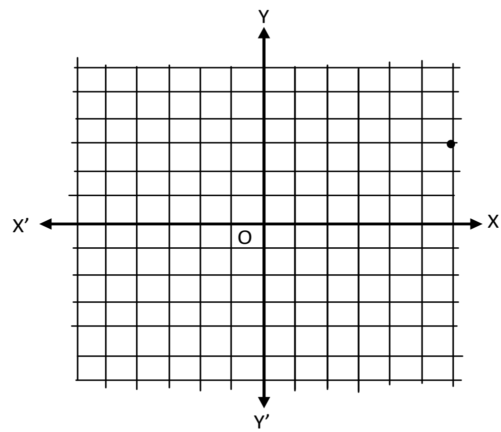
(c) $(2x - 3y, 6) = (7, x + y)$

(d) $\left(4x + \frac{3}{y}, 9\right) = \left(7, 3x + \frac{6}{y}\right)$

5. **Let the number in dice 1 represents first component and the number in dice 2 represents second component. Roll both the dice simultaneously for three times and list the three ordered pairs of numbers. Graph these ordered pairs in the graph paper and analyze your finding as**


(1) (2)

- Three points lying on one line
- Three points making an isosceles triangle
- Three points making a right – angled triangle etc.



1.2 Cartesian product of two sets:

Let A = {Red, Blue} and B = {Pen, Carryon, Pencil}. How many pairs of colored object can be made?

Consider the two sets A = {1, 2} and B = {3, 4}

Now, list the set of ordered pair of numbers (a, b) so that the first component a is the element from the set A and second component b is the element from the set B and denote this set by A × B.

We have, A × B = {(1, 3) , (1, 4) , (2, 3) , (2, 4)}

Here the new set A × B read as "A cross B" is called the cartesian product of the two sets A and B.

If A and B are the two non – empty subset of the universal set U, then the set of all ordered pairs (a, b) such that 'a' is the element from the set A and 'b' is the element from the set B is called the Cartesian Product of the two sets A and B denoted by A × B and read as “A cross B”

In symbol, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

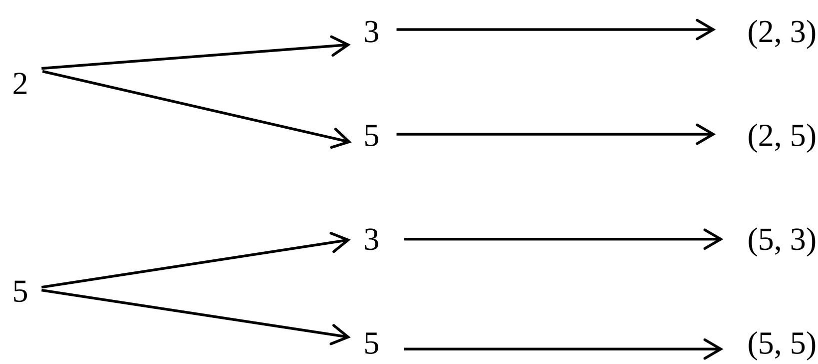
Example 1

Let $A = \{2, 5\}$ and $B = \{3, 5\}$ be the two sets. Find the cartesian product A × B and B × A and check whether A × B = B × A

Solution:

To find A × B, we have

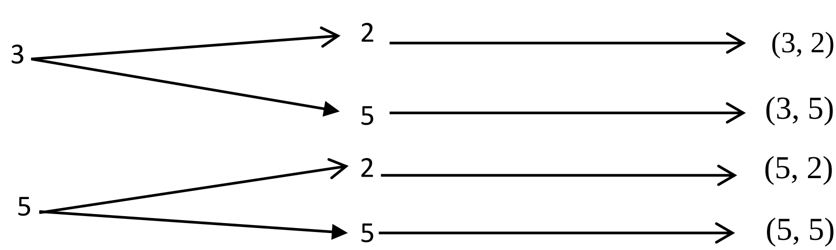
Element of Set A	Paired with elements of set B	Ordered pairs



Hence, $A \times B = \{(2, 3), (2, 5), (5, 3), (5, 5)\}$……………………… (i)

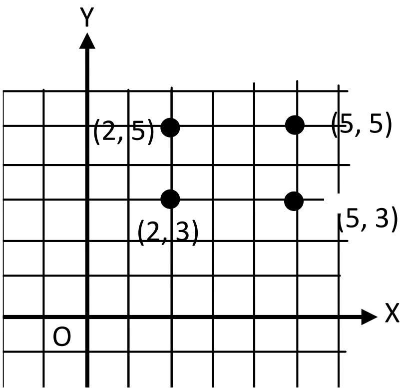
To find B × A, we have

Element of Set B	Paired with elements of set A	Ordered pairs

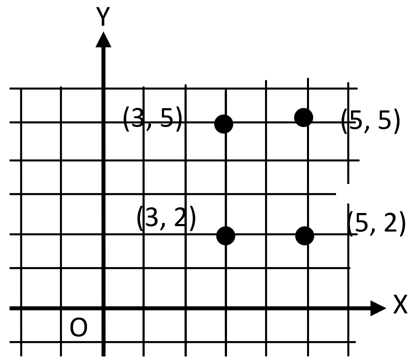


Therefore, $B \times A = \{(3, 2), (3, 5), (5, 2), (5, 5)\}$…………………………(ii)

From (i) and (ii) it is evidenced that A × B ≠ B × A. It could be visualized clearly if we graph these cartesian products as in the following.



A × B



B × A

From the graph it is obvious that A × B ≠ B × A

Method of representing cartesian product

We may represent the cartesian product of two sets by different methods. For example, the two sets A = $\{3, 2\}$ and B = $\{1, 4\}$ then we may represent the cartesian product A × B by any of the following methods:

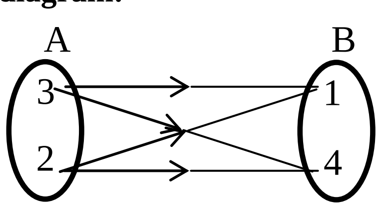
(i) **By listing elements:**

The cartesian product of A and B (A × B) = $\{(3, 1), (3, 4), (2, 1), (2, 4)\}$

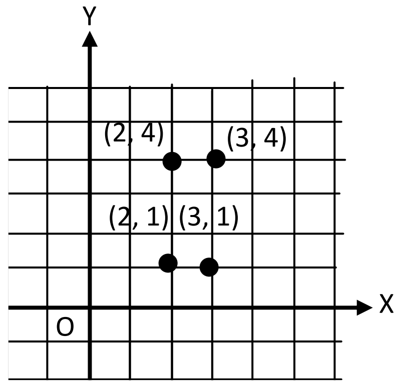
(ii) **By tabulating:**

	Set B		
Set A	×	1	4
	3	(3, 1)	(3, 4)
	2	(2, 1)	(2, 4)

(iii) **By an arrow diagram:**



(iv) **By graph:**



Example 2

Let A = {3, 5} and B = {5, 4} then find A × B and represent by

(i) set of ordered pairs (ii) tabulating

(iii) an arrow diagram (iv) graph.

Solution:

We have,

A = {3, 5} and B = {5, 4}

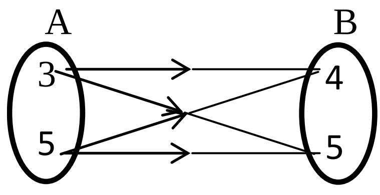
Then

(i) A × B = {(3, 5), (3, 4), (5, 5), (5, 4)}

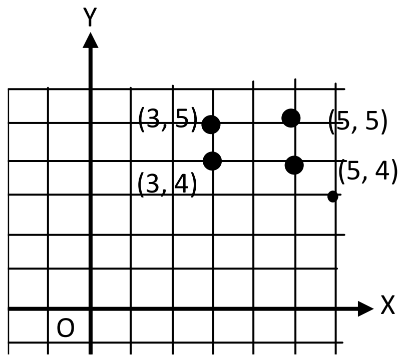
(ii) By tabulating:

A \ B	5	4
×	5	4
3	(3, 5)	(3, 4)
5	(5, 5)	(5, 4)

(iii) By an arrow diagram:



(v) By graph:



Example 3

If A = {1, 2} and B = {3, 5}, find

(a) $A \times B$, n(A), n(B) and $n(A \times B)$ and verify $n(A\times B) = n(A) \times n(B)$

(b) $A \times A$, n(A) and $n(A \times A)$ and verify $n(A\times A) = n(A) \times n(A)$

(c) $B \times B$, n(B) and $n(B \times B)$ and verify $n(B \times B) = n(B) \times n(B)$

Solution:

(a) Here;

A = {1, 2} and B = {3, 5}

Then,

$A \times B = \{(1,3),(1,5),(2,3),(2,5)\}$

And,

n(A) = 2

n (B) = 2

$n(A \times B) = 4 = 2 \times 2 = n(A) \times n(B)$

Therefore, $n(A \times B) = n(A) \times n(B)$

(b) Here,

A = {1, 2}

Then

$A \times A = \{(1,1),(1,2),(2,1),(2,2)\}$

And

n (A) = 2

n (A × A) = 4 = 2 × 2

∴ n (A × A) = n(A) × n(A)

(c) Here, B = {3, 5}

Then, B × B = {(3, 3), (3, 5), (5, 3), (5, 5)}

Also,

n (B) = 2,

n(B × B) = 4 = 2 × 2

= n(B) × n(B)

∴ n(B × B) = n(B) × n(B)

In general, we have the following rules:

(a) If B = A, then A × B = A × A = B × B = B × A

(b) If the cardinalities of the sets A and B are n(A) and n(B) respectively then n(A × B) = n(A) × n(B)

Example 4

If A = $\{x: x \leq 3, x \in N\}$ and B = $\{x: x^2 - 1 = 0\}$, find (a) A × B and n(A × B), (b) A × A and n(A × A).

Solution:

Here, A = $\{x: x < 3, x \in \mathrm{N}\} = \{1, 2, 3\}$

B = $\{x: x^2 - 1 = 0\}$

= $\{-1, 1\}$

Now,

a. We have A × B = $\{(1, -1), (1, 1), (2, -1)(2, 1), (3, -1), (3, 1)\}$

Again,

n(A) = 3, and n(B) = 2

n(A × B) = n(A) × n(B)

= 3 × 2 = 6

b. Here, A = {1, 2, 3}

Then, A × A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

Here, $n(A) = 3$

$n(A \times A) = n(A) \times n(A) = 3 \times 3 = 9$.

Example 5

If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{-1, 4\}$, find.

a. $A \times (B \cap C)$

b. $(A \times B) \cap (A \times C)$

c. Is $A \times (B \cap C) = (A \times B) \cap (A \times C)$?

Solution:

a) Here, $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{-1, 4\}$, then

$B \cap C = \{3, 4\} \cap \{-1, 4\} = \{4\}$

And, $A \times (B \cap C) = \{(1, 4), (2, 4), (3, 4)\}$

b) We have

$A \times B = \{(1, 3), \underline{(1, 4)}, (2, 3), \underline{(2, 4)}, (3, 3), \underline{(3, 4)}\}$………(i)

And $A \times C = \{(1, -1), \underline{(1, 4)}, (2, -1), \underline{(2, 4)}, (3, -1), \underline{(3, 4)}\}$…….(ii)

Now from (i) and (ii) taking the underlined elements

$(A \times B) \cap (A \times C) = \{(1, 4), (2, 4), (3, 4)\}$.

c) From (a) and (b) , we have

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

In general, we have

- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Example 6

If A $\{2, 3, 4\}$, $B = \{3, 4, 5\}$ and $C = \{4, 5\}$, find

a. $A \times (B - C)$

b. $(A \times B) - (A \times C)$

c. Is $A \times (B - C) = (A \times B) - (A \times C)$ True?

Solution:

Here, $A = \{2, 3, 4\}$, $B = \{3, 4, 5\}$ and $C = \{4, 5\}$

Then, $B - C = \{3\}$

Now

a. $A \times (B–C)$

$= \{(2, 3), (3, 3), (4, 3)\}$…………………(i)

b. $(A \times B)$

$= \{(2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5)\}$

$(A \times C) = \{2, 3, 4\} \times \{4, 5\}$

$= \{(2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5)\}$

$\therefore (A \times B) – (A \times C) = \{(2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5)\} – \{(2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5)\}$.

$\therefore (A \times B) – (A \times C) = \{(2, 3), (3, 3), (4, 3)\}$………..(ii)

c. Now from (i) and (ii), we have $A \times (B – C) = (A \times B) – (A \times C)$

In general, if A, B, and C are non – empty subsets of the universal set, we have, $A \times (B – C) = (A \times B) – (A \times C)$

Exercise 1.2

1. a) Define cartesian product with an example.

b) If n(A) = 2 and n(B) = 3, what is the cardinality of $A \times B$?

c) Under what condition $A \times B = B \times A$?

d) What is equal to $A \times (B \cap C)$?

2. If $A = \{1, 2, 3\}$, $B = \{3, –2\}$ find

a) $A \times A$ b) $B \times B$ c) $A \times B$ d) $B \times A$

3. Represent the cartesian product of Q no.2 by

i) Listing elements ii) Tabulation

iii) Arrow diagram iv) Graph

4. If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, find

a) $A \times B$ b) $B \times A$ c) Is $A \times B = B \times A$?

5. a) If $A \times B = \{(a, 1), (a, 5), (a, 2), (b, 2), (b, 1), (b, 5)\}$, find n(A), n(B), and $n(B \times A)$.

b) If $A \times B = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$ find n(A), n(B), and $n(B \times A)$.

6. If A = {1, 2}, B = {1, 2, 3, 4} and C = {5, 6}, verify that

   i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

   iii) $A \times (B \cap C)$ $(A \times B) \cap (A \times C)$ iv) $A \times (B - C) = (A \times B) - (A \times C)$

7. a) If $A = \{x : x \leq 4, x \in N\}$, and $B = \{x : x^2 - 5x - 6 = 0\}$, find A × B and B × A.

   b) If $P = \{2 < x < 7, x \in N\}$ and $Q = \{x : x^2 = 3x\}$, find P × Q and Q × P.

8. a) List the set of alphabets from the word TOKYO and name it T. List the set of alphabets from the word KYOTO and name it K and workout:

      i) T × K (ii) K × T. What is your conclusion?

   b) List the set P from the alphabets of the first name of Ram Bahadur Pun and list the set Q from the alphabets of the last name of Goma Baral and workout: i) P × Q ii) Q × P.

1.3 Relation

There are several examples in day to day life from relation. For example, consider the following ordered pairs.

a) (1, 2) b) (3, 9) c) (Ram, Sita)

With these ordered pairs, we can define the relation of the first elements with second as in the following:

a) (1, 2) gives the relation of first element with respect to the second as 'is half of'.

b) (3, 9) gives the relation 'is square root of'.

c) (Ram, Sita) gives the relation 'is husband of'

If we reverse the order of the elements of the above ordered pairs, we may get different relation as:

a) (2, 1) gives the relation 'is double of'

b) (9, 3) gives the relation 'is square of'

c) (Sita, Ram) gives the relation 'is wife of'

If (a, b) ∈ R, then we define the relation between the ordered pairs (a, b) as aRb.

Now we discuss the following examples.

Example 1

Let A = {1, 2, 3} and B = {1, 4, 9}, then

A × B = {(1, 1), (1, 4), (1, 9), (2, 1), (2, 4), (2, 9), (3, 1), (3, 4), (3, 9)}. Let us define a relation R_1 from A to B by "is square root of".

Then, $1R_11$, $2R_14$, $3R_19$

i.e. R_1 = {(1, 1), (2, 4), (3, 9)}

Let A and B be two non – empty sets, then a relation (R) from set A to the set B is defined as the subset of the Cartesian product A × B.

Symbolically, R = {(a, b) : a ∈ A, b ∈ B and R ⊂ (A × B)}

Example 2

Let A = {1, 2, 3} and B = {1, 2, 3, 4} find

a. A × B
b. Relation from A to B defined by 'is double of'
c. Relation from A to B defined by 'is less than'
d. Relation from A to B defined by 'is square of'
e. Relation from A to B defined by 'is square root of'

Solution:

We have, A = {1, 2, 3} and B = {1, 2, 3, 4}

Now,

a. A × B = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)}
b. R_1 = {(2, 1), (4, 2)} is the relation 'is double of'
c. R_2 = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)} is the relation 'is less than'
d. R_3 = {(1, 1), (4, 2)} is the relation 'is square of'
e. R_4 = {(1, 1), (2, 4)} is the relation 'is square root of'

Example 3

Let A = {2, 3, 4} then find

a. A × A

b. A relation from A to A defined as 'is equal to'

c. A relation A to A defined as 'is greater than'

d. A relation from A to A defined by 'is square root of'

Solution:

Here A = {2, 3, 4}, then

a. A × A = {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

b. R_1 = {(2, 2), (3, 3), (4, 4)} is the relation from A to A defined by 'is equal to'

c. R_2 = {(3, 2), (4, 2), (4, 3}is the relation from A to A defined by 'is greater than'

d. R_3 = {(2, 4)} is the relation from A to A defined by 'is square root of '

Representation of relation:

Consider the two sets A = {1, 3, 5} and B = {2, 4, 6}

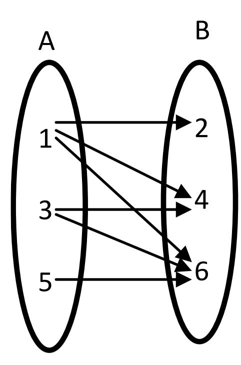
Then A × B = {(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6), (5, 2), (5, 4), (5, 6)} and let R be the relation from A to B defined by 'is less than' which is given by R = {(1, 2), (1, 4), (1, 6), (3, 4), (3, 6), (5, 6)}

Now we can represent the relation R:A→B by any one of the following methods:

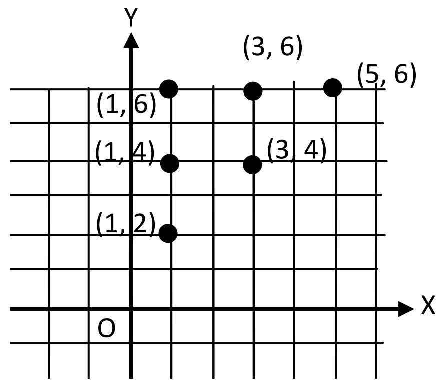
a. By tabulating:

X	1	1	1	3	3	5
Y	2	4	6	4	6	6

b. By an arrow diagram:



c. By graph



d. By a set of ordered pairs:

R = {(1, 2), (1, 4), (1, 6), (3, 4), (3, 6), (5, 6)}

e. By description or formula:

R = {(x, y): x < y, $x \in$ A and y $\in$ B}

Exercise: 1.3(A)

1. a) Define 'relation' and illustrate it.

b) List down the methods of representing a relation.

2. If A × B = {(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)}. Find

a) $R_1 = \{(x, y): x + y = 6\}$ b) $R_2 = \{(x, y): x < y\}$

c) $R_3 = \{(x, y): y = x^2\}$

d) Represent the above relations by

(i) table (ii) arrow diagram (iii) Graph.

3. If A = {1, 3, 5} and B = {1, 3, 6], then find the following relations defined in A × B.

a) is greater than b) is equal to

c) is double of d) is square of

e) Represent the above relation by

i) Set of ordered pairs iv) Graph

ii) Tabulation v) Rule or formula

iii) An arrow diagram

4. If $A = \{6, 7, 8, 10\}$ and $B = \{2, 4, 6\}$ find the following relations defined in $A \times B$.

   a) $R_1 = \{(x, y): (x + y) < 12, x \in A \text{ and } y \in B\}$

   b) $R_2 = \{(x, y): (2x + y) > 0, x \in A \text{ and } y \in B\}$

   c) Represent each of the above relation by means of

      i) set of ordered pairs   ii) by an arrow diagram

      iii) a graph   iv) a table

5. Let $R = \{(1, 2), (2, 3), (3, 4), (5, 6), (6, 7)\}$ be a relation, represent this relation.

   a) by table   c) by graph

   b) by an arrow diagram   d) by description.

6. List the set A that represents your family members. Work out $A \times A$ and list down as many relation as you can from A to A that states the relation between your family members.

Domain and Range of a Function

Consider the two sets $X = \{1, 2, 3\}$ and $Y = \{3, 4, 5\}$ then

$X \times Y = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)\}$

Let a relation from set X to the set Y be defined as $R = \{(x, y): y = x + 2, x \in X \text{ and } y \in Y\}$.

Then $R = \{(1, 3), (2, 4), (3, 5)\}$

Here, the set of first components of the ordered pairs in R can be listed as $R_d = \{1, 2, 3\}$.

Here, R_d is called the domain of the relation R.

Again, the set of second components of all ordered pairs in R can be listed as:

$R_r = \{3, 4, 5\}$.

Here, R_r is called the range of the relation R.

Let A and B be two non – empty sets and R be the relation from A to B then the domain of R is the set of first elements of the ordered pairs in R and range of R is the set of second elements of the ordered pairs in R. Symbolically we write; Domain of R (R_d) = $\{x: (x, y) \in R\}$ and Range of R (R_r) = $\{y: (x, y) \in R\}$

Example 1

Find the domain and range of the following relations:

a. $R_1 = \{(2, 3), (3, 4), (3, 5), (4, 5), (5, 6)\}$

b. $R_2 = \{(1, x), (2, y), (3, z)\}$

Solution:

a. Here, $R_1 = \{(2, 3), (3, 4), (3, 5), (4, 5), (5, 6)\}$ is given

Now, domain of $R_1 = \{x: (x, y) \in R_1\} = \{2, 3, 4, 5\}$

Range of $R_1 = \{y: (x, y) \in R_1\} = \{3, 4, 5, 6\}$

b. Here, $R_2 = \{(1, x), (2, y), (3, z)\}$ is given

Now, domain of $R_2 = \{x: (x, y) \in R_2\} = \{1, 2, 3\}$

Range of $R_2 = \{y: (x, y) \in R_2\} = \{x, y, z\}$.

Types of relation

a) Reflexive relation

A relation R: A→A is reflexive if $(x, x) \in R \leftrightarrow xRx$, $x \in A$ is true. That is a relation on a set A is reflexive if every elements of the set of ordered pair representing the relation is related to itself. For example;

A × A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)} and relation R = {(1, 1), (2, 2), (3, 3)} is a reflexive relation.

b) Symmetric relation

A relation R: A→ A is symmetric if $xRy \rightarrow yRx$. i.e.if $(x, y) \in R$ then $(y, x) \in R$. For example;

A × A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}and a relation R = {(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)} is a symmetric relation.

c) Transitive relation

A relation R: A→A is transitive if xRy and $yRz \Rightarrow xRz$. i.e. if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$. for example, consider A × A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)} then the relation R = {(1, 2), (2, 3), (1, 3)} is a transitive relation.

If a relation satisfies all three above conditions, then it is called equivalence relation. i.e, A relation R is an equivalence relation if it is reflective, symmetric and transitive as well.

Inverse Relation

If R_1:A→B be a relation defined by $R_1 = \{(x, y): x \in A \text{ and } y \in B\}$ and R_2 :B→A be the relation defined as $R_2 = \{(y, x), x \in A \text{ and } y \in B\}$ then R_1 and R_2 are inverse relations to each other. That is a relation obtained by interchanging the order of the ordered pairs in the given relation is inverse to that relation. For example;

$R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and

$R_2 = \{(2, 1), (3, 2), (4, 3)\}$ are inverse relation to each other.

The inverse of any relation R is denoted by R^{-1}.

Example 2

Let A = {4, 5, 6}. Find

a) A × A

b) A relation R_1 in A that is reflexive.

c) A relation R_2 in A that is symmetric.

d) A relation R_3 in A that is transitive.

e) Two relations R_4 and R_5 in A that are inverse to each other.

Solution:

Here, A = {4, 5, 6} Therefore,

a) $A \times A = \{(4, 4), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$

b) $R_1 = \{(x, x): x \in A\} = \{(4, 4), (5, 5), (6, 6)\}$

c) $R_2 = \{(x, y): (x, y) \in R \text{ and } (y, x) \in R\}$

$= \{(4, 5), (5, 4), (5, 6), (6, 5)\}$

d) A set R_3 is transitive in A if $xR_3y \rightarrow yR_3z \rightarrow xR_3z$.

Therefore, $R_3 = \{(4, 5), (5, 6), (4, 6)\}$

e) Let $R_4 = \{(4, 5), (4, 6), (5, 6)\}$ and

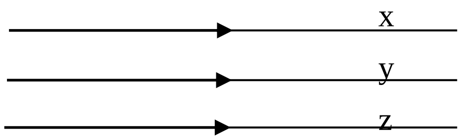
$R_5 = \{(5, 4), (6, 4), (6, 5)\}$ be two relations in A × A, then R_4 and R_5 are inverse relation to each other. This is just an example; there are in fact many relations in A having their inverse relation also defined in A.

Example 3

Let R be the relation defined on a set of all lines in the plane such that R denote the relation "is parallel to" show that R is an equivalence relation.

Solution:

Consider a set of parallel lines,



a) Here $x//x$, y//y and z//z because every line is parallel to itself. Hence xRx → the relation R is Reflexive.

b) We have, x//y → y//x, y//z → z//y, x//z → z//x. Hence. xRy → yRx → the relation R is Symmetric.

c) Here, x//y and y// z → x//z. That is xRy and yRz → xRz, the relation R is Transitive.

Since, relation R is reflexive, symmetric and transitive, so, it is an Equivalence Relation.

Exercise 1.3 (B)

1 (a) Define 'domain' and 'range' of a relation with example.

(b) What is meant by an inverse relation? Give an example.

(c) Define the following relation with an example in each.

(i) Symmetric relation (ii) Reflexive Relation

(iii) Transitive relation (iv) Equivalence relation

2. Find the domain and range of the following relations.

a) {(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)}

b) {(2, 4), (2, 6), (3, 6), (3, 9), (4, 8), (4, 12)}

c) {(5, 8), (6, 9), (7, 10), (8, 11)}

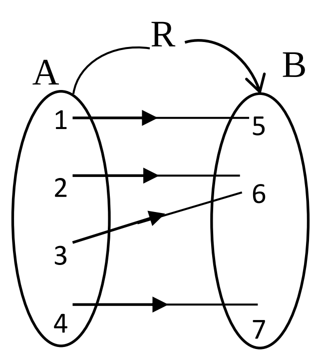
d) {(8, 6), (7, 5), (6, 4), (5, 3), (4, 2), (3, 1)}

3. Find the inverse relation to each relation given in question 2 and state their domain and range.

4. The relation R from A → B is denoted by an arrow diagram

a) Write the relation R as the set of ordered pair of numbers.

b) State the domain and range of the relation.



c) Write down the inverse relation R^{-1} in the form of set of ordered pairs.

5. **If A = {1, 2, – 3, – 4} is the domain of relation, find:**

   a) The range of the relation R_1, if the second element of the ordered pair in R_1 is double of the first element.

   b) Range of the relation R_2, if the second element is one more than the double of the first element in the set of ordered pairs in R_2.

   c) Range of the relation R_3, if the second element of the ordered pairs in R_3 added to the first gives the sum equal to –2.

   d) Range of the relation R_4, if the second element of the ordered pairs in R_4 is equal to 2.

6. **Let A = {2, 3, 4, 5, 6, 7, 8, 9} be the given set and a relation R in A × A is define as R = {(*x*, y) : y is the multiple of *x*}. Write the relation R in the form of ordered pairs and find:**

   a. Domain of R

   b. Range of R

   c. Inverse relation R^{-1}

   d. Domain and range of the inverse relation R^{-1}.

7. **Find the range of the each of the following relations.**

   a. $\{(x, y): y = 2x + 1, 0 \leq x \leq 3, x \in W\}$

   b. $\{(x, y): y = x^2 - 1, 0 \leq x \leq 3, x \in W\}$

   c. $\{(x, y): y = 3x^2 - 2x - 1, 1 \leq x \leq 4, x \in W\}$

   d. $\{(x, y): y = 5^2 - 3x, 0 \leq x \leq 5, x \in W\}$

8. **Find the inverse of each of the following relations.**

   a. {(1, 0), (2, 1), (3, 2), (4, 3)}

   b. {(– 1, – 1), (0, 0), (1, 1), (2, 2)}

   c. {(3, – 1), (4, – 2), (5, – 3), (6, – 4)}

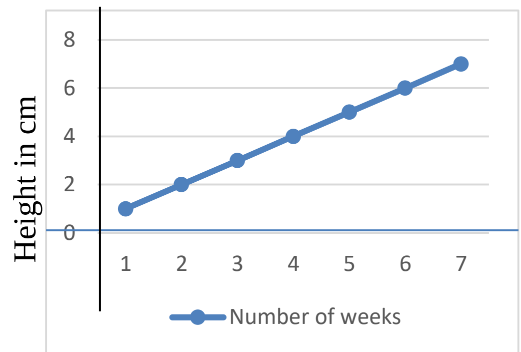
   d. {(4, – 2), (4, 2), (1, – 1), (1, 1)}

9. **Find the inverse of each in the form of set of ordered pairs for each of the relation given in question no.7.**

10. **Identify any two suitable example of daily life which satisfies all conditions of equivalence relation.**

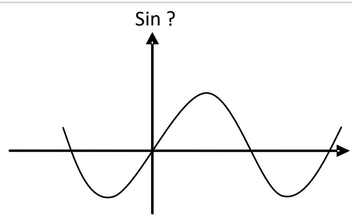
1.4 Functions

Look at the graph alongside. It gives the growth of a plant recorded weekly.



Here, for each number of weeks there corresponds a unique height of the plant. We say that the height increase as the function of time.

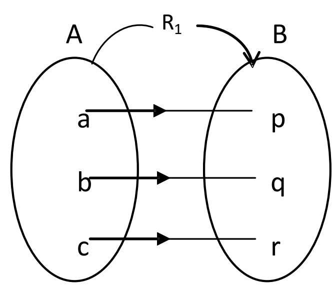
Look at the graph of sine ratio:



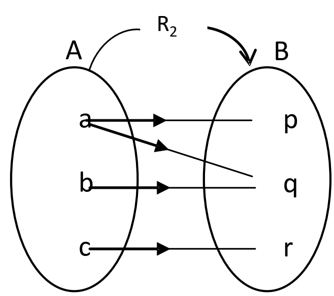
To each θ positive or negative, there corresponds a unique value for sin θ. We say that sine ratio is the function of θ.

Consider the following arrow diagrams

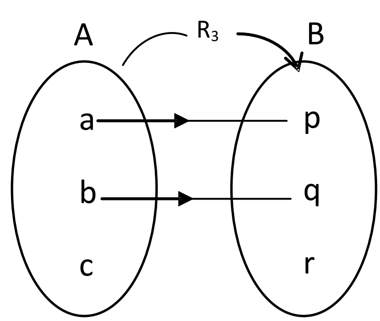
1. Each element of the domain set A has unique image in the range set B. There is one to one pairing between the elements of two sets. We say that R_1 is a function.



2. Here one element a of domain set A is paired with two elements of the range set B. we say that R_2 is not a function. It is simply a relation.



3. Here the element c of the domain set A has no image in the range set B. We say R_3 is not a function. It is simply a relation.



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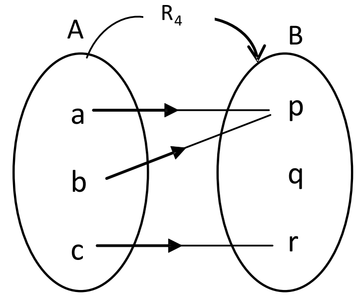
4. Here, however a and b of the domain set A have one image p in the range set B, we say that each element in the domain set has unique image in the range set, and as such R_4 is a function.



Let A and B be two non–empty sets, then a special type of relation f: A→B becomes a function if each element of set A has unique image in set B. If x is an element in set A, then its image in set B is denoted by f(*x*). Symbolically,

$y = f(x)$, $x \in A$, $y \in B$.

Domain and range of a function

Let f:A→B be a function . Then set A is called the domain of the function f and the set B is called the co – domain of the function 'f '. If the elements of set A are denoted by the variable x and their images by the variable y then the subset of the co – domain containing the elements y is called the range of the function f. Here, the element *x* is called the pre – image and y is called the image of x under the function f.

If f:A→B be a function	Domain of f = A
Co – domain of f = B	Range of f = $\{f(x):x \in A\} \subseteq B$

Example 1

Which of the following relations are functions?

a) f = {(3, 2), (4, 1), (5, 0), (4, 2)}

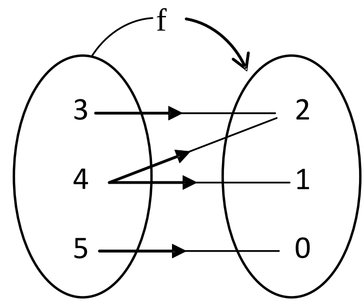
b) g = {(2, 1), (1, 0), (0, – 1), (– 1, – 1)}

c) h = {(3, 2), (4, 2), (5, 2), (6, 7)}

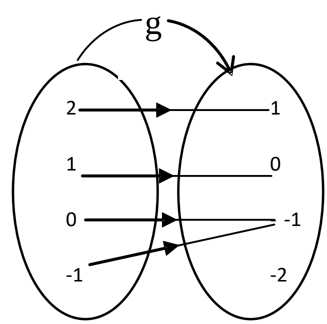
d) k = {(1, a), (2, b), (3, c), (4, d)}

Solution

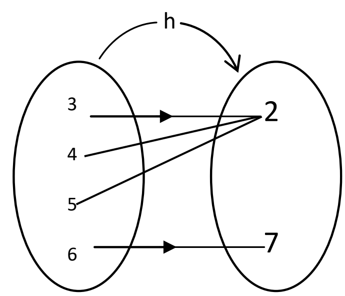
a. In the relation f, there are two ordered pairs (4, 1) and (4, 2) with same pre image 4 and by definition f is not a function. (In a mapping if two images have the same pre – image; it is not a function.)



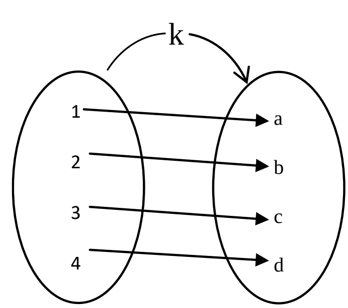
b. In the relation g, each element in the domain set has unique image in the co–domain and by definition g is a function.



c. In the relation h, one image 2 has three pre – images 3, 4 and 5 and 6 has an image 7. We say that each element in the domain has an unique image in the co – domain and by definition h is a function.



d. In the relation K, each element in the domain has unique image in the co – domain, the relation k defines a function.

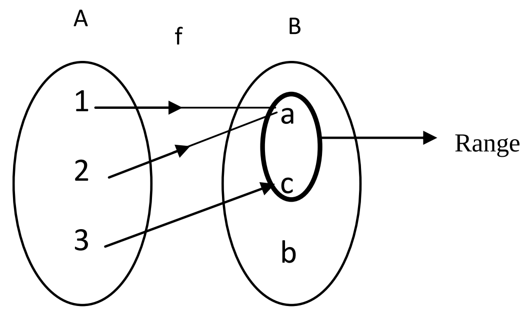


Example 2

Let A = {1, 2, 3} and B = {a, b, c} be given sets and a function f is defined by.

f(1) = a, f(2) = a, f(3) = c

Find the domain, co–domain and range of f.



Solution:

Here, domain of f is A = {1, 2, 3}

Co – domain of f = {a, b, c}

Given that, f(1) = a

f(2) = a

f(3) = c

∴ Range of f = {a, c}

If f:A→B is a function then each element of A must have image in co – domain B but it is not necessary that each element of the co – domain has a pre – image in A.

Exercise – 1.4(A)

1. a) Define a function with an example.

b) What is meant by domain of a function?

c) What is meant by range of a function?

d) What does co – domain mean?

2. Define the following term with example.

a) Image b) pre – image

3. State whether the following relation are function or not.

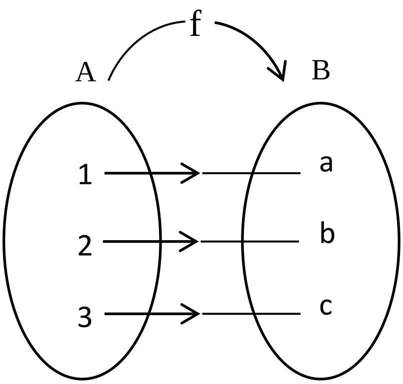
a) f = {(1, 1), (2, 4), (3, 9), (4, 16)}

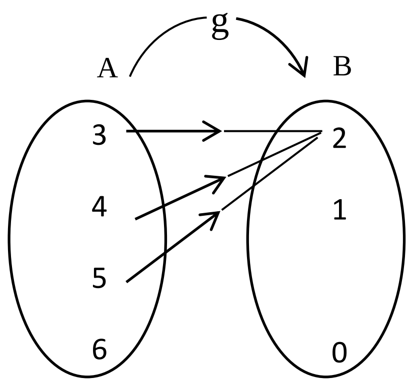
b) g = {(±1, 1), (±2, 4), (±3, 9)}

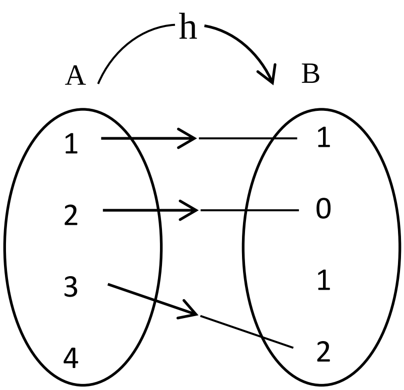
c) h = {(4, 3), (5, 3), (6, 3), (7, 3)}

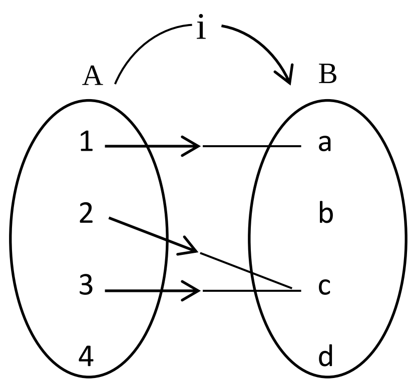
d) i = {(5, 2), (5, 3), (5, 4), (6, 1)}

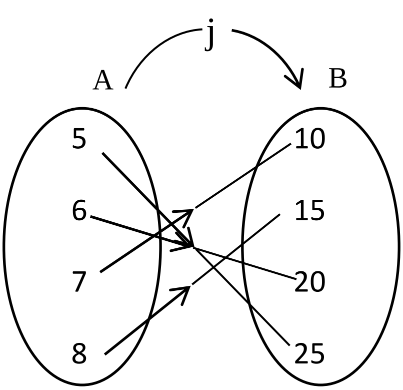
4. State whether the following relation are function or not.

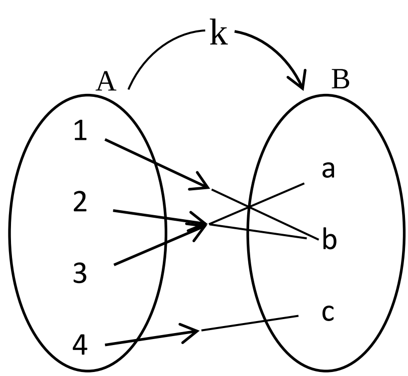












5. **Represent each of the following relation by an arrow diagram and state clearly which of these relation represent the function.**

   a) $\{(1, 2), (3, 6), (-2, -4), (-4, -8)\}$

   b) $\{(-5, 3), (0, 3), (6, 3)\}$

   c) $\{(9, -5), (9, 5), (2, 4)\}$

   d) $\{(-2, 5), (5, 7), (0, 1), (4, -2)\}$

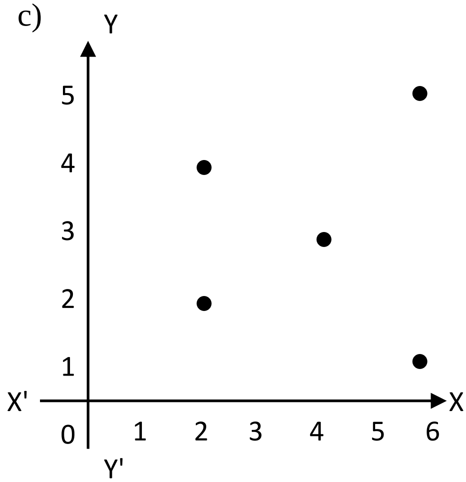
6. **State whether the following relation is a function or not.**

   a)

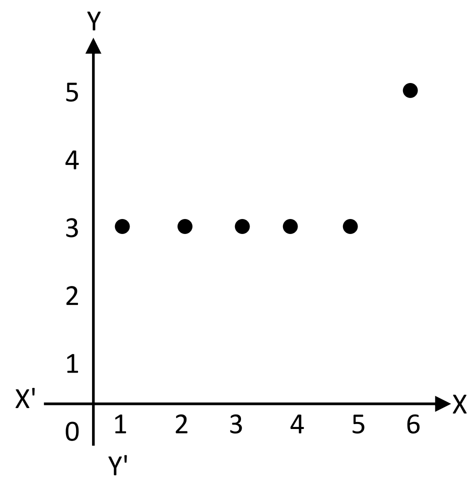
X	3	4	5	6
Y	2	2	1	0

   b) $2 \mapsto 4,\quad 3 \mapsto 2,\quad 4 \mapsto 1,\quad 2 \mapsto 5$

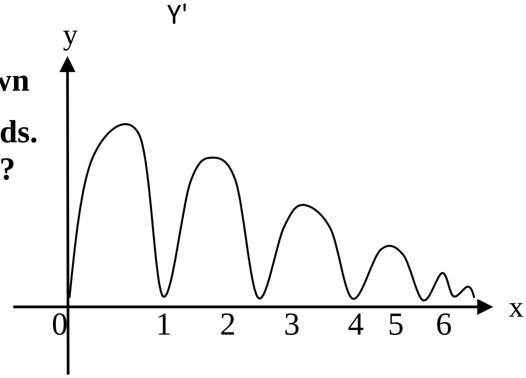
   c)



   d)



7. **Graph shows the bouncing of the lawn tennis ball in the duration of 6 seconds. Is this a relation, function or neither? Give your answer with reason.**

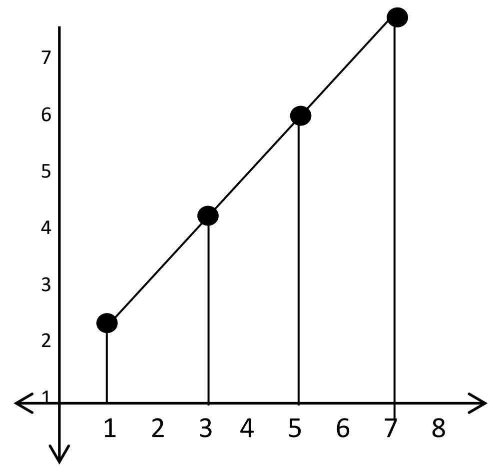


Vertical line Test

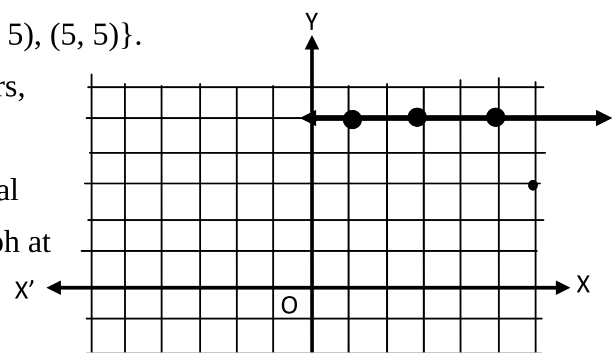
We can define a function as the set of ordered pairs (x, f(x)) where each x is distinct and no two x's will have the same f(x) or y. This concept helps us to test whether a given graph represents a function or not by applying the vertical line test. We can draw several vertical lines perpendicularly to x – axis and check if anyone of them intersects the graph of a function at two or more points. If it is the case the graph doesn't represent a function. Consider the following example:

Example 1

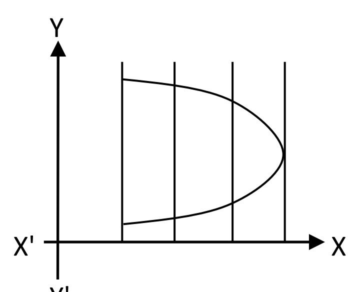
a. Consider f(x) = {(1, 2), (3, 4), (5, 6), (7, 8)}. Graphing these ordered pairs we get, the graph as shown. By drawing several vertical lines perpendicular to x – axis, it can be seen that there is no vertical line that intersects the graph of f(x) at more than one point. We say that the graph represent the function.



b. Consider, g(x) = {(1, 5), (3, 5), (5, 5)}. Graphing these ordered pairs, we get the graph as shown. Here, too, there is no vertical lines that intersects the graph at more than one point. We say that g(x) is a function.



c. Consider h(x) = {(1, 1), (1, 5), (3, 2), (3, 4), (4, 3), (5, 2)}. Graphing these ordered pairs, we get the graph as shown. Did you notice here, the vertical lines are intersecting the graph at two points (more than one point). We say that h(x) is not a function.



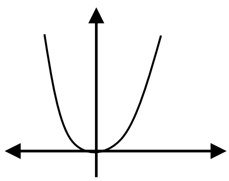
If a vertical line interests the graph of f(x) at only one point then f(x) is a function, otherwise it is not.

Example 2

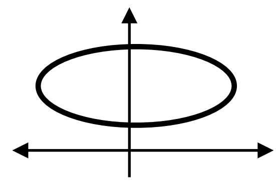
Which of the following graph represents a function?

Solution:

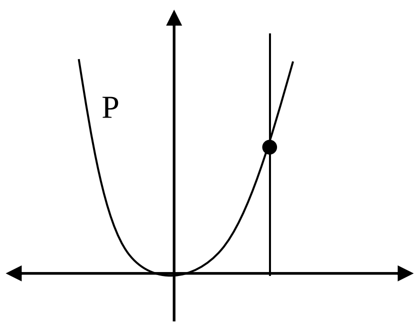
a)

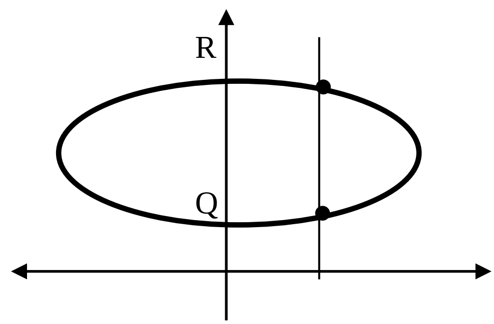


b)



Drawing vertical lines perpendicular to x – axis in each we get,



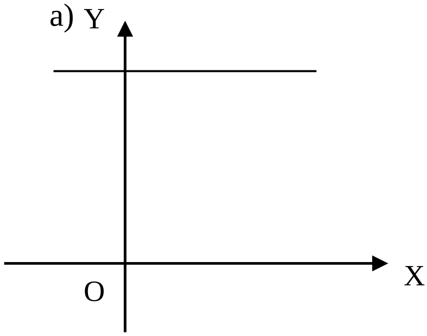


The vertical line intersects the graph in (a) at one point only. Therefore, it is a function. The vertical line intersects the graph in (b) at two points. Therefore, it is not a function.

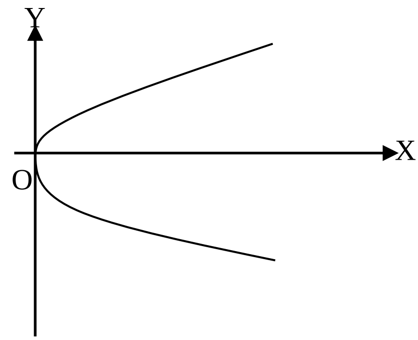
Exercise 1.4(B)

1. Apply vertical line test and determine which of the following graph represents a function.

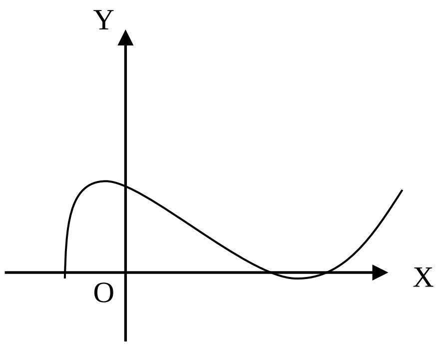
a)



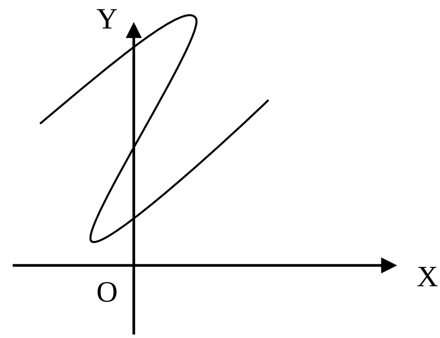
b)



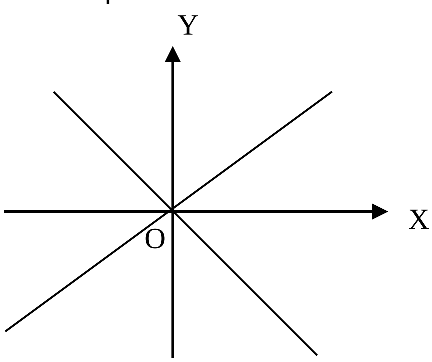
c)



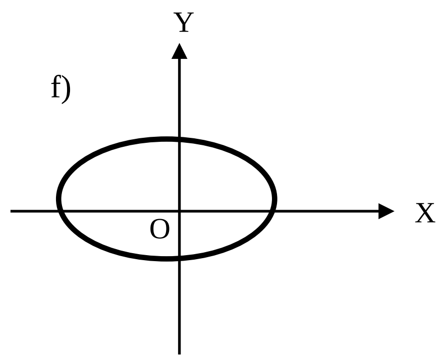
d)



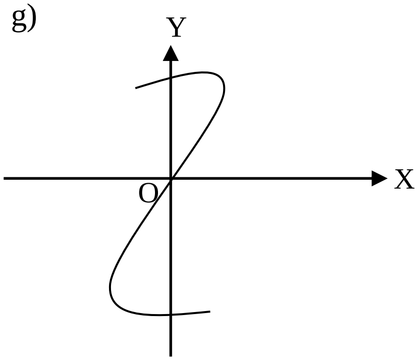
e)



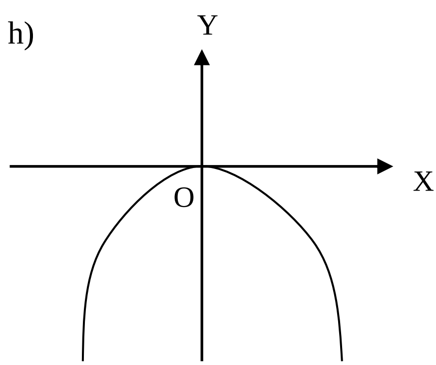
f)



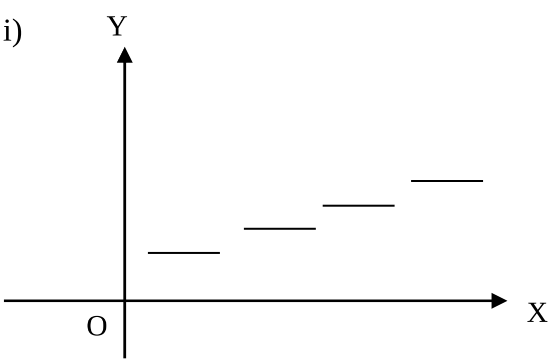
g)



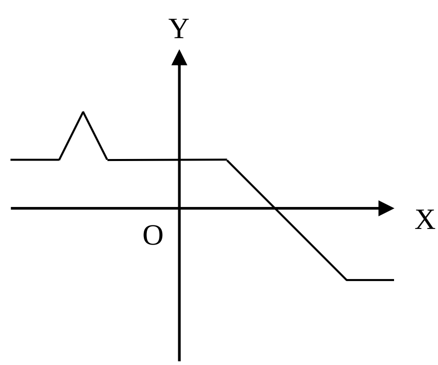
h)



i)



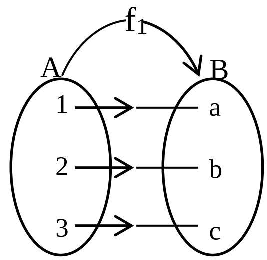
j)

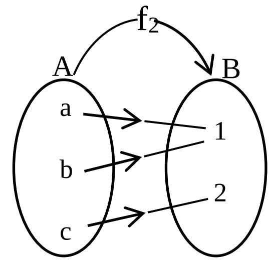


Types of function

a) Onto function

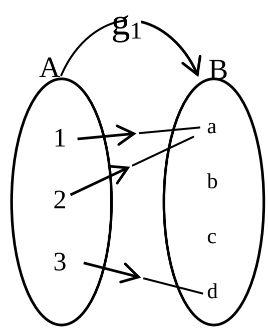
A function f: A→B is an onto function if its range is equal to its co – domain. In the arrow diagram f_1 and f_2 are onto functions.

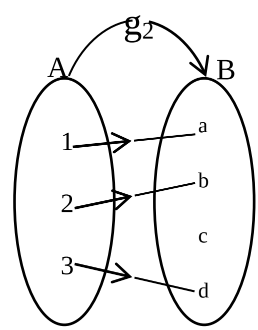




b) Into function

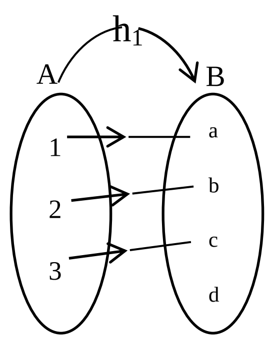
function f:A→B is an into function if its range is proper subset of its co-domain. In the given arrow diagram g_1 and g_2 are into functions.

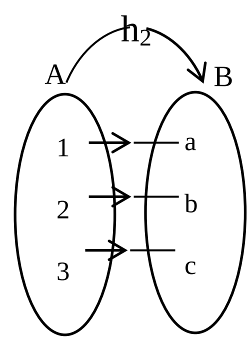




c) One to one function

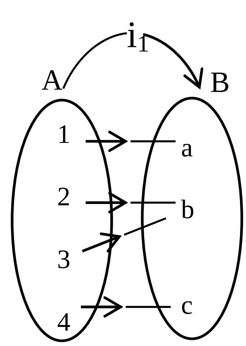
A function f:A→B is an one to one function if each image has an unique pre–image in the domain. In the given arrow diagram, h_1 and h_2 are one to one functions. Moreover, h_1 is one to one and into function, whereas h_2 is one to one and onto function.

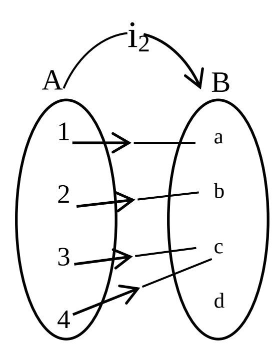




d) Many to one function:

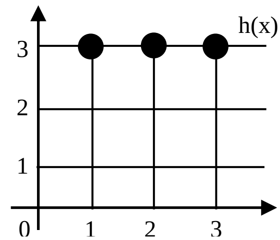
A function f:A→B is many to one function if an image has more than one pre–image in the domain. For example, in the arrow diagram functions i_1 and i_2 are many to one onto and many to one into function respectively.

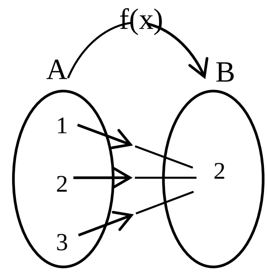




More types of Function

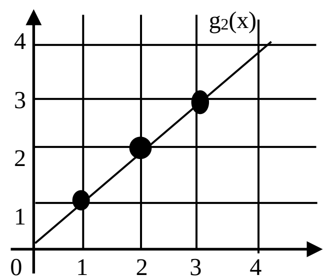
a. Constant function:

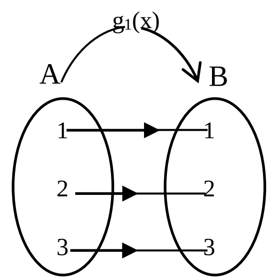




A function f:A→B is a constant function if each element in the domain has only one image in the range. For example, f(x) and h(x) in the given figure are constant functions.

b. Identity function:



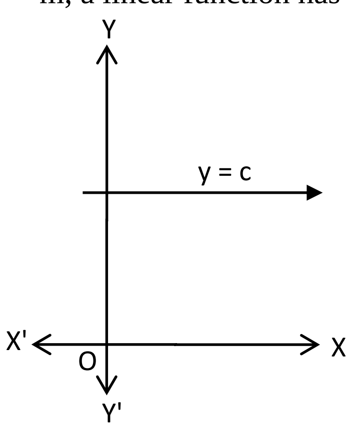


A function f : A → B is an identity function if its image and pre-image are same. In another word, if an element maps onto itself, this type of function is an identity function.

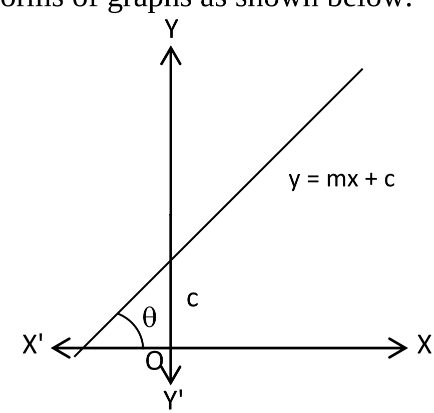
Here, the figure $g_1(x)$ and $g_2(x)$ are identity function.

c. Linear function:

A function of the form f(x) = mx + c, which gives a straight line having slope m and y – intercept c when graphed is a linear function. Based on the value of m, a linear function has different forms of graphs as shown below.



m = 0

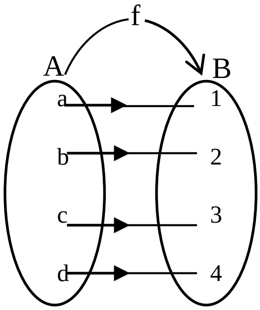


m has definite value

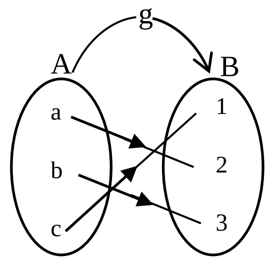
Exercise 1.4 (C)

1. State the kind of function in each of the following.

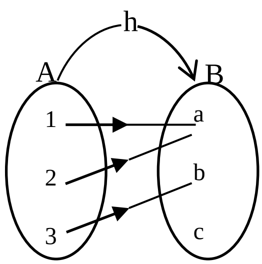
a)



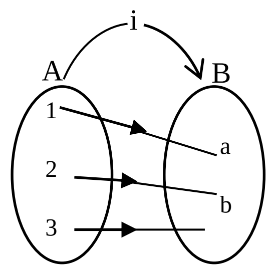
b)



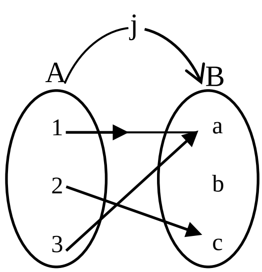
c)



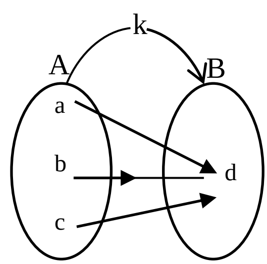
d)



e)

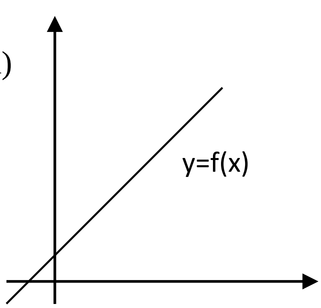


f)

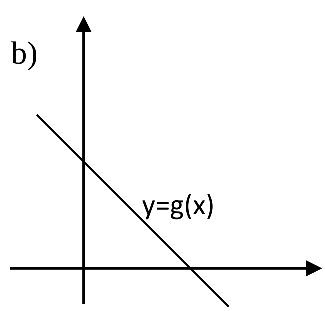


2. State the types of function represented by the following graph.

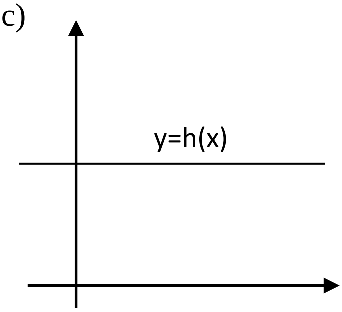
a)



b)



c)



3. **Draw mapping diagram to each function given below and state the type of function.**

a) f = {(1, 2), (2, 4), (3, 6)}

b) g = {(1, 1), (– 1, 1), (2, 4), (3, 9)}

c) h = {(1, 0), (2, 0), (3, 0)}

4. **If A = {1, 2} and B = {p, q, r} how many functions from A→B can be defined which is:**

a) one to one

b) many to one

c) one to one and onto

d) many to one and into.

Values of a function

Let f:A→B be a function that associates $x \in A$ to unique $y \in B$ then y is called the value of the function. It is denoted by $y = f(x)$.

Here, $f(x)$ is the image of x under the function f and x is called the pre – image of $f(x)$ or the pre – image of y.

Example 1

If $f(x) = x^2 + 2$, find the value of f(–1), f(1), f(2) and f(– 2).

Solution:

To find f(–1), we have substitute x = – 1 in $f(x) = x^2 + 2$.

Hence, $f(-1) = (-1)^2 + 2 = 3$.

Likewise,

$f(1) = (1)^2 + 2 = 3$

$f(2) = (2)^2 + 2 = 6$

$f(-2) = (-2)^2 + 2 = 6$

Example: 2

If f:A→B is defined as $f(x) = 2x + 1$ and A = {–1, 0, 1, 2} find the range of f.

Solution:

As the range of f is the set of all images obtained by substituting $x = -1, 0, 1, 2$ in the given function.

Hence, we have,

$f(-1) = 2(-1) + 1 = -1$

$f(0) = 2(0) + 1 = 1$

$f(1) = 2(1) + 1 = 3$

$f(2) = 2(2) + 1 = 5$

Therefore, the range of f is the set {–1, 1, 3, 5}

Example 3

If $f(x) = 3x - 5$ and 7 is one of the image, then find the pre – image of 7.

Solution:

Here, 7 is the image of $f(x)$

$\therefore f(x) = 7$

Or, $3x - 5 = 7$

Or $3x = 12$

$\therefore x = 4$ is the pre – image of 7

Example 4

If $f(x + 2) = 3x - 2$, find f(2)

Solution:

Here, $f(x + 2) = 3x - 2$

Or, $f(x + 2) = 3(x + 2) - 8$

$\therefore f(x) = 3x - 8$

And $f(2) = 3 \times 2 - 8 = -2$

Exercise 1.4(D)

1. (a) If $f(x) = 4x + 5$, find f(2), f(3), f(5).

(b) If $f(x) = 2x^2 - 1$, find f(–1), f(0), f(2).

(c) If $f(x) = 3x^2 + 2x - 1$, find f(0), f(2), f(4)

(d) If $g(x) = x^3 - 2$, find g(1), g(–1), g(2), g(– 2)

2. Find the range of each of the function given below if the domain D is given.

a. $f(x) = 2x - 4$, $D = \{1, 0, 3\}$

b. $g(x) = 3x + 1$, $D = \{1, 3, 5\}$

c. $h(x) = 2 - 3x$ $D = \{-1, 0, 1, 2, 3\}$

d. $k(x) = x^2 + 2$ $D = \{-1, 0, 1, 2\}$

3. (a) If $f(x + 2) = 5x - 8$, find $f(x)$ and f(5)

(b) If $f(x + 1) = 3x + 4$, find $f(x)$ and f(3).

(c) If $f(2x - 1) = 4x + 7$, find $f(x)$ and f(– 2)

(d) If $f(3x + 2) = 12x - 5$, find $f(x)$ and f(6).

4. (a) If $f(x) = x - 5$, find f(h), $f(x + h)$ and $\frac{f(x+h)- f(x)}{h}$ where $h \neq 0$.

(b) If $g(x) = x^2 - 2x$, find g(2), g(2 + h) and $\frac{g(2+h) - g(2)}{h}$ where $h \neq 0$.

(c) If $f(x) = \begin{cases} 3x - 1, x > 0 \\ x + 1, x < 0 \end{cases}$ is a given function, find f(– 1), f(1/5), f(0).

(d) If $f(x) = \begin{cases} 4x - 1 \; if \; -3 < x < 0 \\ 1 + x \; if \; 0 \leq x < 2 \\ x^2 + 9 \; if \; 4 \leq x < 5 \end{cases}$ is a given function, find f(4), f(1)and f(– 2).

5. (a) If P = {0, 1, 2, 3, 4, 5, 6, 7} be a given set and a relation R:P→P is defined as R = {(x, y) :x + y≤7}, find the domain and range of R. Is R a function?

(b) If $f(x) = x^2 - 3$ and one of the image is 22, find the pre – image.

(c) If $g(x) = x^2 - 2x + 1$ and one of the image is 1, find the pre – image.

(d) What is the use of function in your daily life? Discuss in small group with your friends and prepare a report.

1.5 Polynomials

Introduction to Polynomials

We can classify the algebraic expression (or the algebraic function) in the following categories.

(1) On the basis of number of terms in a function

Function	Number of terms	Name
$f(x) = 4x$	1	Monomial
$f(x) = 4x + 5$	2	Binomial
$f(x) = 4x^2 + 4x + 7$	3	Trinomial

(2) On the basis of power (degree) of the variable

Function	Degree	Name of function
$f(x) = 4x + 3$	1	Linear function

$f(x) = 3x^2 + 4x + 5$	2	Quadratic Function
$f(x) = x^{10} + 3x^7 + 12$	10	Polynomial function

Hence, a polynomial might be a monomial, binomial, trinomial etc. based on the number of terms, may be linear, quadratic, cubic or of higher degree based on the highest power (degree) of the polynomials and may be polynomial over integers, over rational number, or over real number according to the nature of coefficients in it.

> The expression of the form $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ $a_n \neq 0$ where $a_1, a_2, a_3 \ldots, a_n$ are real numbers and n is a non – negative integer, is called a polynomial of degree n in x.

In the polynomial function,

$P(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$,

i) If the coefficient $a_0, a_1, a_2, a_3, \ldots\ldots, a_n$ are all positive integers, it is a polynomials of degree n in x over non – negative integers.

ii) If the coefficients $a_0, a_1, a_2, a_3, \ldots\ldots, a_n$ are rational numbers, it is a polynomial of degree n in x over rational numbers.

iii) If the coefficient $a_0, a_1, a_2, a_3, \ldots\ldots, a_n$ are real number, it is a polynomial of degree n in x over real numbers.

For example,

(i) $P(x) = 3x^2 - 2x + 7$ is a polynomial of degree 2 in x over integers.

(ii) $P(x) = 5x^5 - \frac{3}{7}x^4 + \frac{3}{10}x^3 + 7$ is a polynomial of degree 5 in x over rational numbers.

(iii) $P(x) = 7x^4 + \sqrt{2}x^2 + \sqrt{5}x + 6$ is a polynomial of degree 4 in x over real numbers.

Note that, a polynomial may be expressed in the ascending power of the variable or in descending power of the variables as in the following:

(i) $P(x) = x^5 + 2x^4 + 3x^3 - 5x^2 + 2x - 7$ is in descending power of variable x.

(ii) $P(x) = -7 + 2x - 5x^2 + 3x^3 + 2x^4 + x^5$ is in ascending power of variable x.

These are examples of polynomials in standard form.

Example 1

Which of the following expression is a polynomial?

(a) $3x^4 + 2x^3 - \frac{7}{9}x^2 + \sqrt{2}x + 7$

(b) $4x^3 + 3x^{1/2} + 5\sqrt[3]{x} + 9$

Solution:

a) It is a polynomial because the power of the variable is positive integers.

b) It is not a polynomial because the power of the variables in 2nd and 3rd terms are not integers.

Literal coefficient and numerical coefficient

If $2xy + 5$ is a polynomial in x, then y is literal coefficient of x, 2 is numerical coefficient of x and 5 is constant term.

Degree of a polynomial

The degree of a polynomial is the highest power of the variable in it. If a polynomial consists of more than one variable, the degree of such polynomial is highest number obtained by adding the power of these variables in it.

Example 2

Find the degree of the following polynomials:

a) $P(x) = 3x^7 - 4x^3 + 3$

b) $Q(x) = 5x^2y + 7x^3y^3 + 11\,x^4y^3$

Solution:

a) It is a polynomial in x, where the highest power of the variable x is 7. Therefore, it has the degree 7.

b) It is a polynomial in x and y where the highest power of the variables is 4 + 3 = 7. Hence the degree of this polynomial is 7.

Equality of Polynomials

Two polynomials $p(x)$ and $q(x)$ are equal if they have,

- Same degree
- Same number of similar terms
- The terms having same index of x have equal coefficients.

Example 3

Find value of a if p(x) = q(x) where $p(x) = 5x^3 + 7x + 8$, and

$q(x) = 5x^3 + ax + 8$

Solution:

Here, $p(x) = 5x^3 + 7x + 8$, and $q(x) = 5x^3 + ax + 8$,

Since $p(x) = q(x)$ then equating the coefficients of like terms, $a = 7$.

Exercise 1.5 (A)

1. **Define the following terms.**
   a) Polynomial
   b) Polynomials in standard form
   c) Degree of a polynomial
   d) Equality of polynomials
2. **Which of the following function is a polynomial function ?**
   a) $f(x) = 2x + 3$
   b) $g(x) = \sqrt[3]{x} + 3$
   c) $h(x) = 2x^3 + \sqrt[3]{x}$
   d) $i(x) = \frac{1}{4}x^3 - 2x^2 + \frac{5}{7}x + 6$
3. **State the degree of polynomials given in question 2.**
4. **Find the numerical and literal coefficients in each of the following polynomials.**
   a) If $2xy$ is a polynomial in x.
   b) If $2xy$ is a polynomial in y.
   c) If $3x^2y$ is a polynomial in x
   d) If $3x^2y$ is a polynomial in y.
   e) If $\frac{2xy + 5}{8}$ is a polynomial in xy.
5. **Find the degree of each polynomial function given below:**
   a) $f(x) = 2x^2y$
   b) $g(x) = 3xyz^2$

c) $f(x) = 3x^2 – 4x^5 + 2x$.

d) $g(x) = 8x^3 – \sqrt{3}\, x + 4x^4 + 5$

e) $h(xy) = 6x^3y^2 + 7xy^3 7xy^4$

f) $g(xy) = 3x^4y – 5x^2y^5 + xy^3$.

6. Express each of the following polynomials in standard form in i) ascending order ii) descending order.

a) $2x^3 + 5x^2 + 9x^4 + 7x$.

b) $\sqrt{3}x^3 + 7x^2 + 3x^4 + 5$

c) $2x^2 – x + 8 + 3x^3$.

d) $4x^3 + 2x^2 – 3 + 4x^5$

7. Write each of the following polynomials in standard form in ascending order?

a) $f(x) = 5x^4 – 7x^3 + 2x^2 – 8x + 9$

b) $g(x) = \sqrt{25}x^4 – \frac{14}{2}x^3 + 4\frac{1}{2}x^2 – 8x + 9$

c) $h(x) = 3x^3 – x^2 + 7x + 8$

d) $j(x) = \sqrt{9}x^3 – 2x^2 + 7x – 8$

e) $k(x) = 5x^4 – 9x^3 + 2x – 6$

f) $l(x) = 5x^4 + \sqrt[3]{729}x^3 + 2x – 6$

8. If the pair of following polynomial functions are equal, find a and b.

a) $f(x) = 6x^6 – 4x^2 – bx + 8$ and $g(x) = ax^6 – 4x^2 + 2x + 8$

b) $f(x) = 7x^4 – ax^3 + 3x + b$ and $g(x) = 7x^4 + 3x$

c) $f(x) = 19x^5 – 12x^4 + ax + 12$ and $g(x) = bx^5 – 12x^4 + 15x + 12$

d) $f(x) = \sqrt[3]{8}x^4 + ax^3 – 3x – 7$ and $g(x) = bx^4 + 9x^3 + bx – 7$.

Operation on polynomials

In this section we shall discuss on addition, subtraction and multiplication of two or more than two polynomials.

a. Addition and subtraction of polynomials

Consider the following example

Example 4

If $p(x) = 4x^3 - 3x^2 - 7x$ and $q(x) = 2x^2 - 5x + 7$ find,

i. $p(x) + q(x)$

ii. $p(x) - q(x)$

Solution:

Here both polynomials could be expressed as the polynomials of the same degree with zero coefficient for the absent term. Hence, expressing p(x) and q(x) in standard form of the same degree in descending order, we get.

$p(x) = 4x^3 - 3x^2 - 7x + 0$

$q(x) = 0x^3 + 2x^2 - 5x + 7$

Now,

i. $p(x) + q(x) = (4x^3 - 3x^2 - 7x + 0) + (0x^3 + 2x^2 - 5x + 7)$

$= (4 + 0)x^3 + (-3 + 2)x^2 + (-7 - 5)x + (0 + 7)$

$= 4x^3 - x^2 - 12x + 7$

Here, we have added the coefficients of like terms

ii. $p(x) - q(x) = (4x^3 - 3x^2 - 7x + 0) - (0x^3 + 2x^2 - 5x + 7)$

$= (4x^3 - 3x^2 - 7x + 0) + (0x^3 - 2x^2 + 5x - 7)$

$= (4 - 0)x^3 + (-3 - 2)x^2 + (-7 + 5)x + (0 - 7)$

$= 4x^3 - 5x^2 - 2x - 7$

Here we have subtracted the coefficients of like terms.

We can perform the above addition and subtraction without introducing the absent term as in the following.

i. $p(x) + q(x) = (4x^3 - 3x^2 - 7x) + (2x^2 - 5x + 7)$

$= 4x^3 + (-3 + 2)x^2 + (-7 - 5)x + 7$

$= 4x^3 - x^2 - 12x + 7$

Here, we have added the like terms.

ii. $p(x) - q(x) = (4x^3 - 3x^2 - 7x) - (2x^2 - 5x + 7)$

$= (4x^3 - 3x^2 - 7x) + (-2x^2 + 5x - 7)$

$= 4x^3 + (-3 - 2)x^2 + (-7 + 5)x - 7$

$= 4x^3 – 5x^2 – 2x – 7$

We can add or subtract two polynomials by adding or subtracting the coefficients like terms.

b. Multiplication of polynomials

Consider the following example:

If $f(x) = x^2 + 2x – 1$ and $g(x) = x^2 – x + 5$, find $f(x) \times g(x)$

Here,

$f(x) = x^2 + 2x – 1$ and $g(x) = x^2 – x + 5$

Now,

$f(x) \times g(x) = (x^2 + 2x – 1) \times (x^2 – x + 5)$

$= x^2(x^2 – x + 5) + 2x(x^2 – x + 5) – 1(x^2 – x + 5)$

$= x^4 – x^3 + 5x^2 + 2x^3 – 2x^2 + 10x – x^2 + x – 5$

$= x^4 + x^3 + 2x^2 + 11x – 5$

We can multiply two polynomials by multiplying each term of the multiplicand polynomials by each term of the multiplier polynomial.

Example 1

If $f(x) = 2x – 1$, $g(x) = 2x + 1$ and $h(x) = 5x^2 + 6x + 2$, find $f(x) \times g(x) + h(x)$

Solution:

Here,

$f(x) \times g(x) = (2x – 1), (2x + 1)$

$= 2x(2x + 1) – 1(2x + 1)$

$= 4x^2 + 2x – 2x – 1$

$\therefore f(x) \times g(x) = 4x^2 – 1$

Now,

$f(x) \times g(x) + h(x) = (4x^2 – 1) + (5x^2 + 6x + 2)$

$= (4 + 5)x^2 + 6x + (– 1 + 2)$

$\therefore f(x) \times g(x) + h(x) = 9x^2 + 6x + 1$

Example 2

If $f(x) = 5x + 1$, $g(x) = 25x^2 – 5x + 1$ and $h(x) = 128x^3 – 4x^2 + 6x + 9$, find $h(x) – f(x) \times g(x)$.

Solution:

Here,

$f(x) \times g(x) = (5x + 1), (25x^2 – 5x + 1)$

$= 5x(25x^2 – 5x + 1) + 1(25x^2 – 5x + 1)$

$= 125x^3 – 25x^2 + 5x + 25x^2 – 5x + 1$

$\therefore f(x) \times g(x) = 125x^3 + 1.$

Now,

$h(x) – f(x) \times g(x) = (128x^3 – 4x^2 + 6x + 9) – (125x^3 + 1)$

$= (128x^3 – 4x^2 + 6x + 9) + (– 125x^3 – 1)$

$= (128 – 125)x^3 – 4x^2 + 6x + (9 – 1)$

$\therefore h(x) – f(x) \times g(x) = 3x^3 – 4x^2 + 6x + 8$

Exercise – 1.5 (B)

1. **Find $f(x) + g(x)$ in each of the following.**
   a. $f(x) = 3x^3 – 4x^2 + 5x – 7$
      $g(x) = 2x^2 – 3x + x^3$
   b. $f(x) = 7x^3 + 4x^2 – 5$
      $g(x) = x^3 – x^2 + 1$
   c. $f(x) = 9x^4 + 8x^3 + 7x – 15$
      $g(x) = 11x^4 – 8x^2 – 12$
2. **Find $f(x) – g(x)$ in each of the polynomials given in question no. 1.**
3. **Verify: $f(x) + g(x) = g(x) + f(x)$, from the polynomials given in question no. 1.**
4. **Find $f(x) \times g(x)$ and $g(x) \times f(x)$ in each of the following.**
   a. $f(x) = (x^3 – 1)$, $g(x) = x^3 + 1$.
   b. $f(x) = (x^2 – x + 1)$, $g(x) = (x + 1)$

c. $f(x) = x^3 - 2x^2 + x - 1$, $g(x) = x^2 - 2x + 4$

d. $f(x) = x^2 - 2x + 1$, $g(x) = (x^3 + 7x^2 - 5)$

5. Verify f(x) x g(x) = g(x) x f(x) using the results in question 4.

6. If $f(x) = 2x^2 + 7$, $g(x) = 3x - 9$ and $h(x) = 5x^2 + 7x - 9$ find

a. $[f(x) \times g(x)] \times h(x)$

b. $f(x) \times [g(x) \times h(x)]$

c. $f(x) \times [g(x) + h(x)]$

d. $g(x) \times [f(x) - h(x)]$

7. Based on the result of question no. 6, verify that (a) = (b)

8. If $f(x) = \frac{2}{3}x^3 + 5 - \frac{1}{9}x$, $g(x) = \frac{1}{2}x^2 + \frac{7}{9}x - 6$ and $h(x) = x^2 + x$. find

a. $[f(x) + g(x)] + h(x)$

b. $f(x) + [g(x) + h(x)]$

c. $[f(x) - g(x)] + h(x)$

d. $f(x) - [g(x) - h(x)]$

9. If $p(x) = 2x^2 + 3$, $q(x) = 3x^2 + x + 1$and $r(x) = 5x + 7$ Verify the following:

a) $p(x) + q(x) = q(x) + p(x)$

b) $[p(x) + q(x)] + r(x) = p(x) + [q(x) + r(x)]$

c) $p(x) \times q(x) = q(x) \times p(x)$

d) $p(x)[q(x) + r(x)] = p(x) \times q(x) + p(x) \times r(x)$

e) $p(x)[q(x) - r(x)] = p(x) \times q(x) - p(x) \times r(x)$

10. a) What must be added to $3x^2 + x + 1$ to make it $5x^2 + 7x - 15$?

b) What must be added to $5x^3 - 7x + 13$ to make it $5x^3 + 7x^2 + 3x - 17$?

c) What must be subtracted from $x^3 + 3x^2y - 4xy^2$ to make it $5x^3 + 7x^2y + 12xy^2$?

d) What must be subtracted from $x^3 - y^3$ to make it $x^3 + 2xy^2 - 3x^2y + 4y^3$?

1.6 Sequence and Series

a) Sequence:

Consider the dot pattern for numbers, see the numbers they represent and continue to the next two terms.

: :: ::: ::::

2 4 6 8

Here, the dot pattern represents the arrangement of number in the definite pattern as 2, 4, 6, 8, which can be continued to the desired number of terms and therefore the next two numbers are 10 and 12. We say that this is an arrangement of rule that the number in desired term is obtained by multiplying the number of term by 2, giving

$2 \times 1 = 2$ first term of the sequence

$2 \times 2 = 4$ second term of the sequence

$2 \times 3 = 6$ third term of the sequence

$2 \times 4 = 8$ fourth term of the sequence

$2 \times 5 = 10$ the desired terms to be continued

$2 \times 6 = 12$ the next desired term to be continued.

In general, the n^{th} term of this sequence is given by $t_n = 2n$, by giving different values to n, we can get the desired terms of the sequence.

A sequence is an ordered set of numbers each of whose term is governed by a fixed rule.

Once we have the general rule, the formula for the n^{th} term, we can play different games with sequences as in the following.

Consider for example

$t_n = 2n$ implies 2, 4, 6, 8, 10, … as the sequence of even numbers.

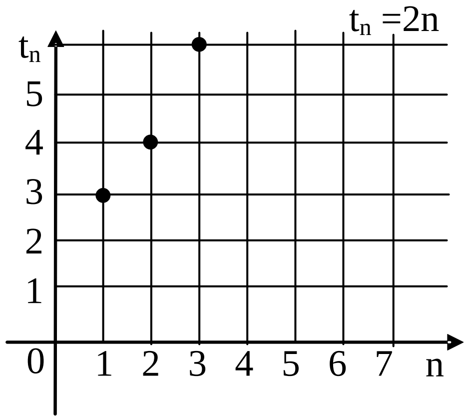
Subtracting 1, from the rule, we get:

$t_n = 2n - 1$ implies 1, 3, 5, 7, 9, … as the sequence of odd numbers.

Adding 1 in the formula, we get:

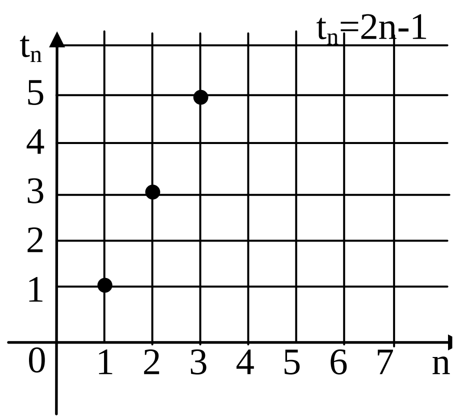
$t_n = 2n + 1$implies 3, 5, 7, 9, … as the sequence of odd numbers greater than 2.

These different kinds of sequence can be graphed by plotting t_n against n as in the following:



It seems that all the points representing the terms of a sequence lie in a straight line, as such these are linear sequences.

Similarly, the quadratic and cubic sequences are generalized.



In fact, a sequence is a function whose domain is the set of natural numbers and the range is subset of the real number.

Example 1

What are the next two terms in the given sequences?

a. 1, 2, 3, 4, 5, …?

b. 2, 5, 8, 11, 14, … ?

Solution:

a. Here the sequence is 1, 2, 3, 4, 5, …, it is seen that each term in the sequence is one more than the immediate preceding term. Therefore, the next two terms are, 5 + 1 = 6 and 6 + 1 = 7. In the general sense, it is a sequence of natural numbers whose n^{th} term is $t_n = n$ and the desired terms are the 6^{th} and the 7^{th} term.

$\therefore t_n = n$

$\rightarrow t_6 = 6$

$t_7 = 7$

b. 2, 5, 8, 11, 14, … is the given sequence; each of the term is 3 more than the immediate preceding term. Hence, the required next two terms are 14 + 3 = 17 and 17 + 3 = 20. In general, the rule gives $t_n = 3n - 1$ as the n^{th} term of the sequence.

$\therefore$ The required next two terms, 6^{th} and 7^{th} terms are:

$t_6 = 3 \times 6 - 1 = 17$

$t_7 = 3 \times 7 - 1 = 20$

Example 2

If $f(n) = 75 + 5n$, $n \in N$ is a rule for the n^{th} term of the sequence, find the first four terms and write them as a sequence of numbers.

Solution:

We have,

$f(n) = 75 + 5n$.

Giving value to n such that $1 \leq n \leq 4$, we get.

$f(1) = 75 + 5 \times 1 = 80$

$f(2) = 75 + 5 \times 2 = 85$

$(3) = 75 + 5 \times 3 = 90$

$f(4) = 75 + 5 \times 4 = 95$

Hence, the required first four terms are 80, 85, 90, 95 and the sequence corresponding to these terms is 80, 85, 90, 95.

Exercise – 1.6 (A)

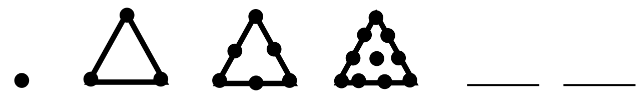
1. **Find the next two terms of the sequences:**
   a. 3, 5, 7, 9, …
   b. – 4, – 2, 0, 2, …
   c. 18, 14, 10, 6, 2, …
   d. 20, 15, 10, 5, …
   e. 5, 10, 20, 40, …
   f. 64, 32, 16, 8, …
2. **By drawing graph of t_n against n for the above sequences, decide whether they are linear sequences or not.**

3. Look at the following patterns of dots that represent sequence of numbers.

a. Find two more patterns.

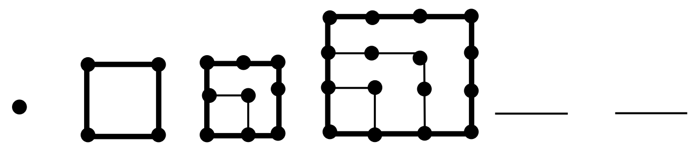
b. Draw graph and decide whether the following sequences are linear, quadratic or cubic sequences.

i.



1 3 6 10 ? ?

ii.



1 4 9 16 ? ?

4. Find the first five terms of the following function and write them as sequence where n is the natural number.

a. $f(n) = 3n + 2$ b. $f(n) = n^2 - 1$

c. $f(n) = 2^n$ d. $f(n) = (-1)^n \times n^2$

5. Consider the n^{th} term of the sequence $t_n = n^2$.

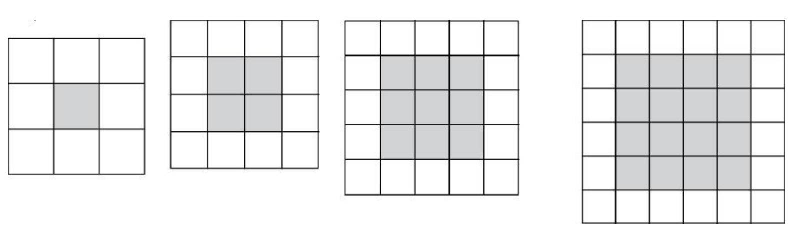
a. Find first six terms of the sequence.

b. In the formula $t_n = n^2$, subtract 1 and find the first six terms of this sequence.

c. In the formula $t_n = n^2$, multiply by 2 and subtract 3, and find the first six terms of the sequences.

6. **The adjoining figure gives the tiling structure of a house; where the shaded part is red tile and the rest is white.**

P indicates the order of the structure of the tiles.



Find the formula for each structure to find the number of tiles needed for the red tile.

a. Find the formula for each structure to find the number of tiles needed for the white tile.

b. Write the formula in terms of p for the total number of tiles?

3.2 The General terms of a sequence:

Consider the set of natural numbers 1, 2, 3, 4, …..n. Tabulate them and try to find the n^{th} term under these operation.

Natural number	1	2	3	4	5	.	.	.	n	t_n
Multiply by 2	2	4	6	8	10	.	.	.	2n	2n
Multiply by 2 and subtract 1	1	3	5	7	9	.	.	.	2n – 1	2n – 1
Square it	1	4	9	16	25	.	.	.	n^2	n^2
Square it and add 2	3	6	11	18	27	.	.	.	$n^2 + 2$	$n^2 + 2$

Now take any one n^{th} term from the table, for example $t_n = 2n - 1$ and substitute n = 1, 2, 3, ….n etc. from natural number you get the terms as

$t_1 = 2 \times 1 - 1 = 1$

$t_2 = 2 \times 2 - 1 = 3$

$t_3 = 2 \times 3 - 1 = 5$

$t_4 = 2 \times 4 - 1 = 7$

$t_5 = 2 \times 5 - 1 = 9$ and so on.

We get the same sequence 1, 3, 5, 7, 9….. as in the table. We say that $t_n = 2n - 1$ is the generator of the terms of a sequence. In the language of sequence, it is called the general term.

To find the general term of a sequence means to find its n^{th} term t_n expressed in terms of n. When given different values to n from the natural numbers, it generates the desired terms of a sequence and the sequence up to desired number of terms.

Method of finding general term of sequence

a) Hit and trial method.

By guessing and testing different values for the common pattern and generate the general term of sequences. For instance, find the n^{th} term of the sequence 1, 4, 9, 16, 25, ………

Make a guess; and rewrite the terms as

$1 = 1\times1 = 1^2 \rightarrow 1^{st}$ term $\quad$ $4 = 2 \times 2 = 2^2 \rightarrow 2^{nd}$ term

$9 = 3 \times 3 = 3^2 \rightarrow 3^{rd}$ term $\quad$ $16 = 4 \times 4 = 4^2 \rightarrow 4^{th}$ term

$25 = 5 \times 5 = 5^2 \rightarrow 5^{th}$ term

Therefore, n^{th} term $=$ (number of term)$^2 = n \times n$

i.e. $t_n = n^2$.

In this method we identify the common pattern among the terms and then generalize for the n^{th} term.

b. Term difference method

In the previous lesson we have been introduced that $t_n = dn + c$ is a linear sequence. Where d is the common difference and c is constant.

Giving different values 1, 2, 3, 4, … to n from the set of natural numbers we have the sequence as.

$d + c$, $2d + c$ $\quad$ $3d + c$ $\quad$ $4d + c$ $\quad$ $5d + c$, …

d $\quad$ d $\quad$ d $\quad$ d $\qquad$ Working out the difference

Here, the first difference is d. A sequence of this type has the general term $t_n = dn + c$, and is a linear sequence.

Example 1

Find the n^{th} term of the sequence, 18, 14, 10, 6, 2, …

Solution:

working out the difference

18 $\quad$ 14 $\quad$ 10 $\quad$ 6 $\quad$ 2

-4 $\quad$ -4 $\quad$ -4 $\quad$ -4 $\qquad$ (first difference)

As the first difference is the same number – 4, its n^{th} term has the form $t_n = dn + c$; where $d = -4$ and given that:

$d + c = 18$

or, $-4 + c = 18$

$\therefore c = 22$

Hence, $t_n = dn + c$

$\rightarrow t_n = -4n + 22$

This is a linear sequence.

ii) Again, we know that $t_n = an^2 + bn + c$ is a quadratic sequence. Substituting

n = 1, 2, 3….. from the set of natural number, we get the sequence as

$a + b + c \quad 4a + 2b + c \quad 9a + 3b + c \quad 16a + 4b + c$

$3a + b \quad 5a + b \quad 7a + b$ (first difference)

$2a \quad 2a$ (second difference)

Here, the second difference is 2a, and the sequence having the second difference constant is a quadratic sequence that has the n^{th} term $t_n = an^2 + bn + c$, in which 1^{st} difference has the first term $3a + b$, second difference is $2a$ and the first term is $a + b + c$.

Example 2

Find the n^{th} term of the sequence. 3, 6, 11, 18, 27, 38, …

Solution:

Sequence 3 6 11 18 27 38 …

3 5 7 9 11 (first difference)

2 2 2 2 (second difference)

As the second difference is a constant number 2, the sequence is a quadratic sequence and has the n^{th} term as $t_n = an^2 + bn + c$.

Here, second difference 2a = 2

∴ a = 1

The first term of first difference 3a + b = 3.

or, 3 × 1 + b = 3.

b = 0

The first term of the sequence

a + b + c = 3

or, 1 + 0 + c = 3

∴ c = 2

Hence, the sequence has the general term

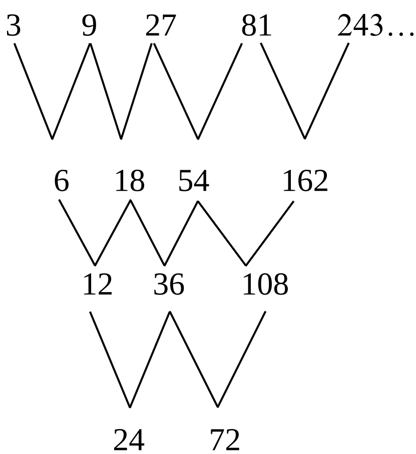
$t_n = an^2 + bn + c.$

or, $t_n = 1(n)^2 + 0(n) + 2$

or, $t_n = n^2 + 2.$

c. Sequence that does not turn into the same difference.

Consider the sequence 3, 9, 27, 81, 243, … Working out the difference.



It seems that the sequence does not turn into the same difference. In such situation we have to go for hit and trial method as sited earlier.

We have,

No of term (n)	1	2	3	4	5
Value of the term	$3 = 3^1$	$9 = 3^2$	$27 = 3^3$	$81 = 3^4$	$243 = 3^5$

Hence, $t_n = 3^n$; $n \in N$

Example 5

Find the n^th^ term of the sequence: $\frac{1}{4}, \frac{4}{9}, \frac{9}{16}, \frac{15}{25}, \ldots$

Solution:

It seems somehow different than the one we have discussed earlier. Hence, we may go far hit and trial method.

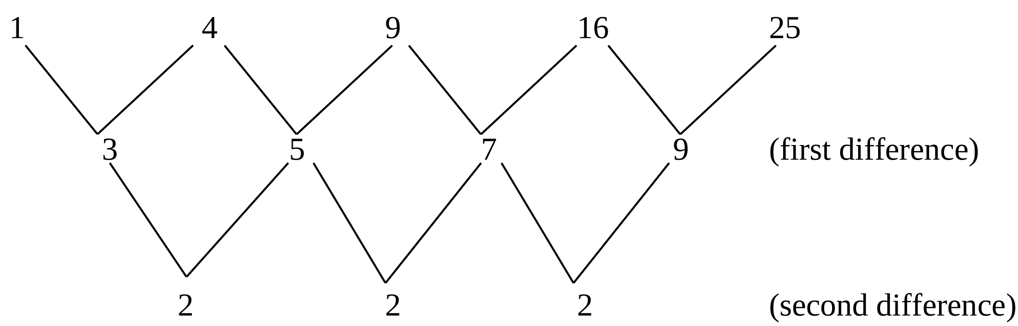
No of term	1	2	3	4	n^{th}
Value of the term	$\frac{1}{4} = (\frac{1}{2})^2$ $= \left(\frac{1}{1+1}\right)^2$	$\frac{4}{9} = \left(\frac{2}{3}\right)^2$ $= \left(\frac{2}{2+1}\right)^2$	$\frac{9}{16} = \left(\frac{3}{4}\right)^2$ $= \left(\frac{3}{3+1}\right)^2$	$\frac{16}{25} = \left(\frac{4}{5}\right)^2$ $= \left(\frac{4}{4+1}\right)^2$	$\left(\frac{n}{n+1}\right)^2$

Hence, the general term (t_n)= $\left(\frac{n}{n+1}\right)^2$

Alternatively,

Take the numerator only, it is 1, 4, 9, 16, 25, …

Working out with the difference



It is quadratic sequence, where

The second difference $2a = 2 \rightarrow a = 1$

The first term of first difference $3a + b = 3$ or, $3(1) + b = 3$ gives $b = 0$.

And the first term $a + b + c = 1$.

or, $1 + 0 + c = 1$ gives $c = 0$

And

or, $t_n = an^2 + bn + c$

or, $t_n = 1n^2 + 0n + 0$

or, $t_n = n^2$.

Taking the denominator and working out the difference we get,

4 9 16 25 36

5 7 9 11 (first difference)

2 2 2 (second difference)

Here, 2a = 2 or a = 1

3a + b = 5 or 3(1) + b = 5 $\therefore$b = 2

a + b + c = 4 or 1 + 2 + c = 4 $\therefore$c = 1 and

or, $t_n = an^2 + bn + c$

or, $t_n = 1n^2 + 2n + 1$

or, $t_n = (n + 1)^2$

Now combining the numerator and denominator, we get.

$t_n = \frac{n^2}{(n+1)^2}$ or, $t_n = \left(\frac{n}{n+1}\right)^2$

Example 4

Find the n^{th} term of the sequence.0, $\frac{-1}{2}, \frac{2}{3}, \frac{-3}{2}, \frac{4}{5}, \frac{-5}{6}$……..

Solution

The sequence may be written as

$\frac{0}{1}, -\frac{1}{2}, \frac{2}{3}, \frac{-3}{4}, \frac{4}{5}, \frac{-5}{6}$, ………….

This is an alternative sequence, where the terms come + , – , + , – , + , – , …

For the sequence of this type the n^{th} term is multiplied by $(-1)^{n+1}$ why?

Taking only positive values and making a table, we get:

No of terms	1	2	3	4	n
Value of the term	$0 = \frac{0}{1} = \frac{1-1}{1}$	$\frac{1}{2} = \frac{2-1}{2}$	$\frac{2}{3} = \frac{3-1}{3}$	$\frac{3}{4} = \frac{4-1}{4}$	$\frac{n-1}{n}$

Hence from the table, $t_n = (-1)^{n+1}\left(\frac{n-1}{n}\right)$

For the alternative method:

Take the numerator sequence as 0, 1, 2, 3, 4, …….

And denominator sequence as 1, 2, 3, 4, 5, ……

And work out the difference method as in Example 3

Exercise – 1.6 (B)

1. Find the first five terms of the sequence if its n^{th} term is given.

a. $t_n = 3n + 1$ b. $t_n = n^2 + 4 + 5$ c. $t_n = 3n^2 - 5$ d. $t_n = n^3 - 3$

2. Find the general term of the following sequences.

a. 5, 7, 9, 11, 13, ……..

b. 5, 2, – 1, – 3, – 7……..

c. 7, 11, 15, 19, 23……

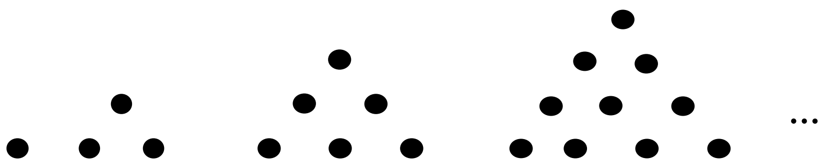
d. 2, 6, 12, 20, 30, …..

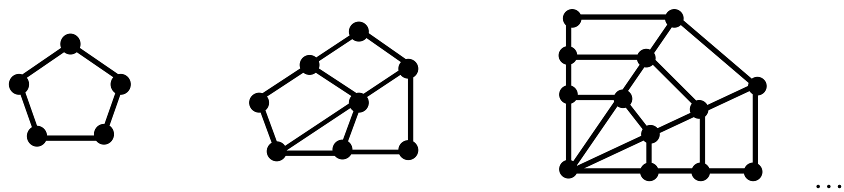
e. $\frac{1}{3}, \frac{4}{5}, 1, \frac{10}{9}$, ….

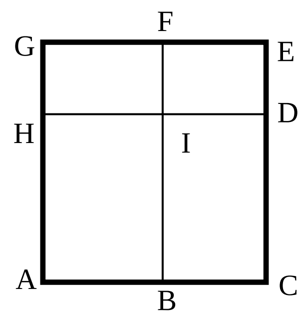
f. $\frac{2}{7}, \frac{5}{8}, \frac{8}{9}, \frac{11}{10}$……….

3. Find the n^{th} term of the following patterns of numbers.

a.	b.	c.
2	4	2
7	16	5
12	36	8
….	64	11
….	100	….
	144	….
	….	….
	….	….

c. 

d. 



4. In the given figure $(n + 10)^2 = n^2 + 20n + 100$. When $n = 1, 2, 3, 4, \dots$, how does the area of the rectangle and square change.

Introduction to Series and Sigma (Σ) Notation:

In everyday speech, the words sequence and series are often used synonymously, however in mathematics they have different meaning. We have discussed earlier about sequence. There is situation we have to sum up the series up to the desired number of terms like as,

$S_1 = 1$ (summing the first term)

$S_2 = 1 + 2$ (summing up to first two terms)

$S_3 = 1 + 2 + 3$ (summing up to first three terms)

$S_4 = 1 + 2 + 3 + 4$ (Summing up to the first four terms) and so on.

These expression giving the partial sums of the terms of a sequence up to desired number of terms are all series.

The sum of the terms of any types of sequence is called the series. i.e if t_1, t_2, t_3, t_4, ... be a sequence then $t_1 + t_2 + t_3 + t_4 + \dots$ is series.

Sigma notation:

One of the effective way to represent the partial sum of the series is by means of sigma notation (Σ) where a gives the starting term and b gives the ending term of the series to be added.

Example 7

Let 1, 3, 5, 7, 9, …(2n – 1) be a sequence. Represent the following partial sums using sigma (Σ) notation.

$S_1 = 1$

$S_2 = 1 + 3$

$S_3 = 1 + 3 + 5$

$S_4 = 1 + 3 + 5 + 7$

$S_5 = 1 + 3 + 5 + 7 + 9$.

To express any partial sum into sigma notation is to find the general term of the sequence that corresponds the partial sums. Here the general term is $t_n = (2n – 1)$. With this general term we express the above partial sums as.

$S_1 = 1 = \sum_{n=1}(2n-1)$ has one term only.

$S_2 = 1 + 3 = \sum_{n=1}^{2}(2n-1)$ has two terms to be added.

$S_3 = 1 + 3 + 5 = \sum_{n=1}^{3}(2n-1)$ has three terms to be added.

$S_4 = 1 + 3 + 5 + 7 = \sum_{n=1}^{4}(2n-1)$ has four terms to be added.

$S_5 = 1 + 3 + 5 + 7 + 9 = \sum_{n=1}^{5}(2n-1)$ has five terms to be added.

Example 8

Expand and evaluate the sum $\sum_{n=1}^{5}(3n + 2)$.

Solution:

Here $\sum_{n=1}^{5}(3n + 2)$ is given. Giving the value of n from 1 to 5 in (3n + 2) we get.

When, n = 1, (3n + 2) = 3 × 1 + 2 = 5

When, n = 2, (3n + 2) = 3 × 2 + 2 = 8

When, n = 3, (3n + 2) = 3 × 3 + 2 = 11

When, n = 4, (3n + 2) = 3 × 4 + 2 = 14.

When, n = 5, (3n + 2) = 3 × 5 + 2 = 17.

$\therefore \sum_{n=1}^{5}(3n + 2) = 5 + 8 + 11 + 14 + 17 = 55$

The symbol sigma Σ stand to mean the sum of all term between a and b.

Example 9

Express the series 1 + 8 + 17 + 28 + 42 in ∑ notation.

Solution:

Let the sequence corresponding to the sum 1 + 8 + 17 + 28 + 42 be 1, 8, 17, 28, 42. Working out the difference, we get.

1 8 17 28 42

7 9 11 13 (first difference)

2 2 2 (second difference)

As the second difference is the constant number 2, the sequence is quadratic sequence with n^{th} term $t_n = an^2 + bn + c$.

Here,

2a = 2

or a = 1.

3a + b = 7

or 3 × 1 + b = 7

or b = 4.

a + b + c = 1

or 1 + 4 + c = 1

or c = – 4

Hence,

$t_n = an^2 + bn + c$

or, $t_n = 1(n)^2 + 4(n) - 4$

$\therefore t_n = n^2 + 4n - 4$.

Therefore, 1 + 8 + 17 + 28 + 42 = $\sum_{n=1}^{5}(n^2 + 4n - 4)$

Exercise – 1.6 (C)

1. (a) Define a series with suitable example.

(b) Differentiate between sequence and series.

(c) What does $\sum$ notation mean?

(d) How many terms shall be added when $\sum_{n=3}^{10} tn$ is given?

2. Which of the following are sequences and which are series?

a) 4, 5, 6, 7, 8, 9, 11

b) $\frac{4}{3}, \frac{5}{3}, \frac{6}{3}, \frac{7}{3}$

c) $\sum_{n=1}^{5} 2n + 3$

d){2n + 5}

e) {(1, 5), (2, 7), (3, 9), (4, 11)}

f) 4 + 7 + 10 + 11 + …

g) 2 + 4 + 8 + 16 + …

3. Express in $\sum$ notation for the following series:

a) 2 + 5 + 8 + 11 + 14 + 17 + 20

b) – 1 + 2 – 3 + 4 – 5 + 6 – 7

c) (a – 1), $(a - 2)^2$, $(a - 3)^2$, …….$(a - 14)^{14}$

d) $\frac{6}{2} + \frac{10}{2} + \frac{15}{2} + \frac{21}{2} + \frac{28}{2}$ + ……. up to 10 terms

4. Find the value of

a) $\sum_{n=1}^{3} 3n$

b) $\sum_{n=1}^{4}(3n - 1)$

c) $\sum_{n=3}^{6}(n^2 + 1)$

d) $\sum_{n=1}^{4} \frac{2n-1}{2n+1}$

e) $\sum_{n=3}^{8}(-1)^n(2n + 1)$

f)$\sum_{n=3}^{8}(-1)^n(2n^2 + 3n - 3)$

5. (a) Find the total amount due at the end of 10th year on the sum of Rs. 25, 000 invested at the rate of 8% p.a simple interest by using $\sum$ notation.

(b) Find the total amount due at the end of 12th year on the sum of Rs. 20, 000 invested at the rate of 10% p.a simple interest by using $\sum$ notation.

Unit 2

Concept of Limit

2.0 Review

Discuss on the following problems in small groups of students.

i) What is function? What will be the value of f(5) if f $(x) = 5x^2 + 8$?

ii) What shape will be formed when the number of sides of a regular polygon increases infinitely?

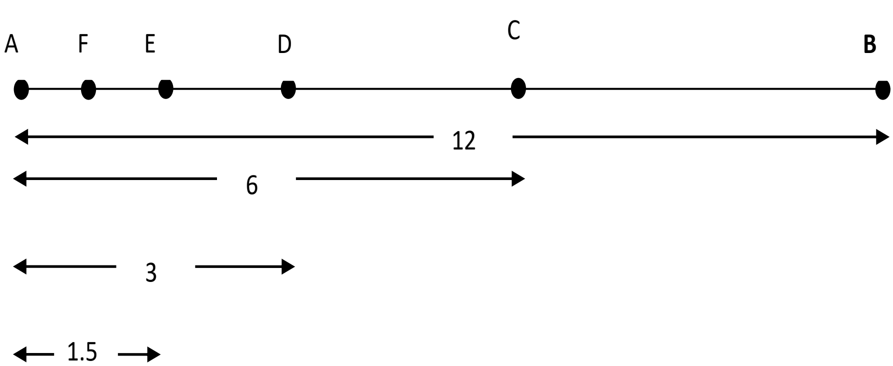
iii) May you complete the running course in the given condition that "you are allowed to run half the distance between end point and your position at a time?

iv) What will be the value of $\frac{1}{n^2}$ when n = 10?

v) Two persons are walking towards a point form opposite side. Will they coincide exactly?

2.1 Limit of Number Sequence

Take a line segment of length 12 cm. Mark at C such that C is the midpoint of AB. Again divide AC into two equal parts with point D. Continuing the same process, divide the segment AD into two halves. What will be the length of the part of line segment We can write it as 12, 6, 3, 1.5, 0.75, 0.375, ………. and so on.



Observe the sequence of number. This sequence is decreasing sequence. Here when the number of piece of line segment increase the length of line segments decreases.

If we further divide into finite parts, what will happen? Discuss.

At the time the length of the piece of segment approaches to zero (but not zero) Then 0 (zero) is limit value of number sequence 6, 3, 1.5, 0.75, 0.375, … In other words, the limit of a sequence of numbers is the value at which the sequence seems to be terminated but not exactly terminated.

Example 1

(a) What will be 7^{th} term of the sequence of 81, 27, 9, 3, ... ?

Solution: Here,

The first term = 81, second term = 27, third term = 9, fourth term = 3 and so on.

ratio of second and first term = $\frac{27}{81} = \frac{1}{3}$

ratio of third and second term = $\frac{9}{27} = \frac{1}{3}$

Here, the ratio of two successive numbers is equal.

So the next term = $\frac{1}{3}$ times of preceding term.

Hence, fourth term = $9 \times \frac{1}{3} = 3$

Fifth term = $3 \times \frac{1}{3} = 1$

Sixth term = $1 \times \frac{1}{3} = \frac{1}{3}$

Seventh term = $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

(b) Is '0' is limit value of this sequence?

Solution: Here,

When the number of term increasing the value of term is decreasing by the ratio of $\frac{1}{3}$. So, when n approaches to infinitely large value (∞) the value of sequence approaches to zero. So '0' is limit of given sequence.

Exercise 2.1

1. The sequence of number is given as 10,1, $\frac{1}{10}, \frac{1}{100}, \ldots\ldots\ldots\ldots$.

a) Find 6th term.

b) At which value will this sequence approaches when the number of term increases?

c) What will be the value of the term when the value of n approaches to infinity?

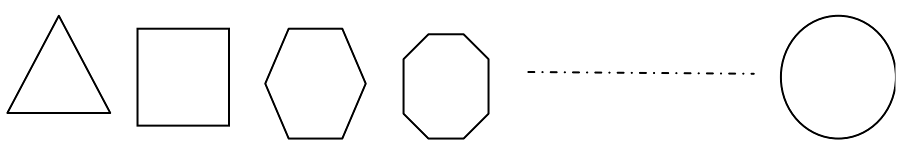
2. a) What will be the 8th terms of sequence 5.01, 5.001, 5.0001, ……….. ?

b) What will be the terminating value of this sequence when the number of the terms approaches to infinity?

3. Take a line segment of length 1 foot. Divide it into two equal half and again divide that one half into two equal parts. What will be the length of the piece of line segment if the same process continues up to 10th time. Show your work in number line, number sequence of length of the line segments.

2.2 Limits from Geometric Figures

Observe the following geometric shapes and discuss on the given questions.



(a) Which regular polygon will form by minimum numbers of line segment?

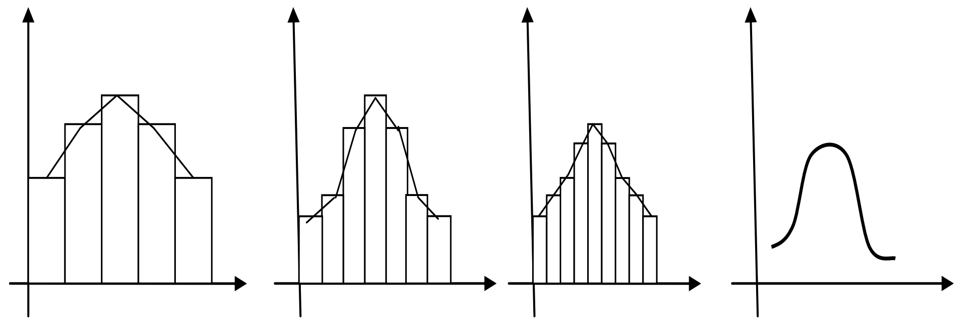
(b) Which geometric figure will form if we increase the number of sides of regular polygon infinitely?

(c) What is the relationship between the number of sides and the geometric figure formed as (b)?

Here, the minimum number of sides required for a closed figure is 3.So equilateral triangle is a regular polygon having minimum number of equal sides. When the number of sides increases we get regular polygon like square, pentagon, hexagon, heptagon, octagon, … and so on. So we get a sequence of polygons with number of sides as Equilateral triangle, square, pentagon, hexagon, heptagon, octagon and so on. If we increase number of sides successively we get a polygon having infinitely small length of side.

i.e. If we continuously increase the number of sides the geometric figure approaches to the circle. So the limiting value of regular polygon is circle.

Example 1



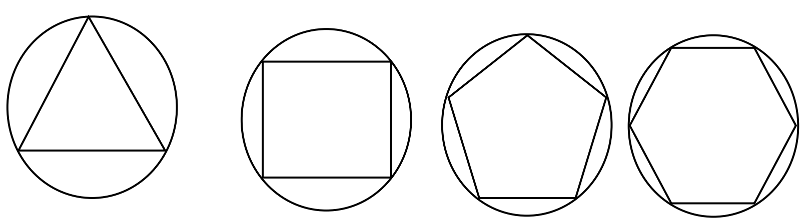
When the length of the intervals in histogram decreases the number of intervals increases. When the number of intervals in data set increases, then the length of line segments of frequency polygon decreases and the number of line segment increases. If the number of intervals increases with the given range of data, the frequency polygon tends to be a curve. This curve is called the frequency distribution curve.

The process of making size small is called binning and the intervals such formed are called bins.

The limiting position of frequency polygon is frequency curve of distribution curve. i.e, if bins become small in length the frequency polygon becomes a curve.

Exercise 2.2

1. Observe the following figures and solve the given question .



a) What will be the trend of difference in area of regular polygon and circle?

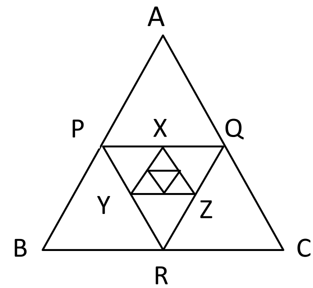
b) What is the trend of difference between the perimeter and circumference of circle?

c) What is the limiting value of difference in areas and perimeter of polygon and circle?

2. Observe the pattern of triangles formed and solve the following question.

Here ΔABC is an equilateral triangle.

P, Q, R be midpoint of AB, BC and AC respectively and continuously.



a) If the area of ΔABC is 10 sq. meter. What will be the area of ΔPQR and ΔXYZ?

b) If we continuously divide the triangle what will be the sequence of area of triangle?

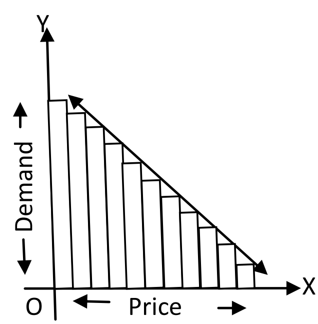
c) When the above process goes infinitely, what will be the limiting value of area of triangle? Estimate.

3. Draw a circle with suitable radius. Draw another circle with same center and radius half of first circle. Similarly draw another circle with same center and radius half then the second circle. Continuing the same process and draw 10 circle.

a) What will be the sequence of areas of these circles?

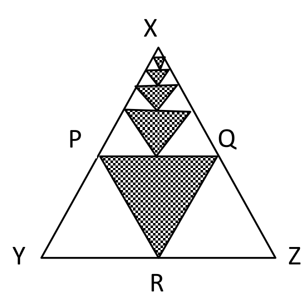
b) What will be the limiting value of area of these circles?

4. The line in the given figure is the demand curve of the commodity when the price increases the demand decreases. What will be the value of demand (Y) if price tends to ∞ in X?



5. What will be the remaining amount after 20[th] division if you have to divide Rs. 2,621,440?

6. The figure given aside ΔXYZ is an equilateral triangle with length of a side '*l*'. If the area of ΔPQR is $\frac{1}{4}$ of ΔXYZ and similar to other shaded small triangles. Find the area of shaded region of the given figure and complete the following.



Triangle	Area	Remark
ΔXYZ		
ΔPQR		

What will be the area if the side of the triangle is infinitely small?

2.3 Limit as sum of infinite series

Let us start from an example of infinite series.

Consider an infinite series; $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ldots\ldots\ldots\ldots$

What will be the sum of the above series? Will it be equal to 2 or more? By taking sufficiently large number of terms of series we can say the sum approach near to 2.

Let S_n be the sum of n-terms of the series, So,

$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ldots\ldots\ldots\ldots$ up to n terms

$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots\ldots\ldots$up to n power of 2

For n = 4 there are 5 terms such as $S_5 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$ and sum is

$S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1 + \frac{8+4+2+1}{16} = 1 + \frac{15}{16} = 1 + 0.937 = 1.937$

which is very close to 2. So, for sufficient large n, the sum is 2. i.e, when n approaches to infinity, S_n approaches to 2

i.e. the limiting value of S_n is 2.

Example 1

Suppose the area of equilateral ΔABC is 1 sq unit P, Q and R be mid points of AB, BC and AC. What will be the area of shaded part of the figure?

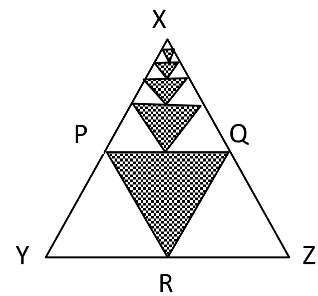
Solution: Here,

Triangle ABC is an equilateral triangle with area = 1 sq. unit

Now, the first shaded part is $\frac{1}{4}$ of 1 = $1\times\frac{1}{4}$ sq. unit = $\frac{1}{4}$ sq. unit

The second shaded part is $\frac{1}{4}\left(\frac{1}{4}\right) = \frac{1}{16}$

The third shaded part is $\frac{1}{4}\left(\frac{1}{16}\right) = \frac{1}{64}$ and so on.



So, $S_n = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots\ldots$ up to n terms

If n = 6, $S_6 = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024}$

$= \frac{1}{4} + \frac{64 + 16 + 4 + 2}{1024}$

$= \frac{1}{4} + \frac{86}{1024}$ which is close to 0.333333.

So, when n is sufficiently large, S_n approaches to 0.33. Hence the limit of the sums of the shaded part is $0.33 = \frac{1}{3}$.

Example 2

Write the first 6 terms and find its limit of $t_n = n^2-1$.

Solution: Here,

The general term is $(t_n) = n^2-1$

so the first six terms are:

when n = 1 , $t_1 = 1^2 - 1 = 0$

when n = 2 , $t_2 = 2^2 - 1 = 3$

when n = 3 , $t_3 = 3^2 - 1 = 8$

when n = 4 , $t_4 = 4^2 - 1 = 15$

when n = 5 , $t_5 = 5^2 - 1 = 24$

when n = 6 , $t_6 = 6^2 - 1 = 35$

Continuing the same process,

when n approaches to 20, the value of t_n approaches to $t_{20} = 20^2 - 1 = 399$.

Exercise 2.3

1. **Find sum of first five terms of the following series and estimate limiting value of the following series for sufficiently large value of n.**

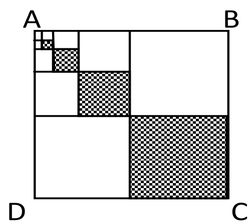
   (a) $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+$ ……………..

   (b) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$ ……………….

   (c) $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots$ … … …

   (d) 0.5 + 0.005 + 0.0005 + ……………….

2. **If the area of the given square ABCD in adjoining figure is 1 square unit, estimate the sum of total shaded parts for sufficiently large n.**



3. **The sequences and their general terms are given below. Find first 6 terms of each of the sequence and find their limit.**

   a) $t_n = 2^n$ b) $t_n = n^2 + n$ c) $t_n = \frac{1}{n^2}$

   d) $t_n = (-1)^n n$ e) $t_n = (-1)^n + n + 1$

2.4 Limit of a Functions

Observe the following example,

Let us consider a function $f(x) = 2x + 1$.
Find the values of $f(x)$ at $x = 1,2$ and 3.

We have, f (1) = 2 x 1 + 1 = 3

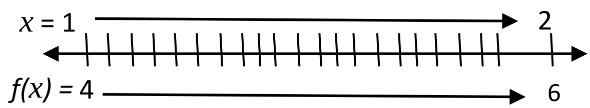
f (2) = 2 x 2 + 1 = 5

f (3) = 2 x 3 + 1 = 7

Also complete the following table for the function f(x) = 2x + 2

x	1	1.2	1.4	1.6	1.8	1.9	1.99	1.998	……	2
y = f(x)	4	4.4	4.8	5.2	5.6	5.8	5.98	5.998	……	6

By using number line we can see that, x approaches to (→) 2 f(x) approaches to (→) 6



Here, when x approaches to 2 the value of f (x) approaches to 6 and when value of $x = 2$ the value of $f(x) = 6$.

Hence, 6 is limiting value of function $f(x) = 2x + 2$ when x approaches to 2. Symbolically we can write limit $x \to 2$ implies $f(x) = (2x + 2) \to 6$

i.e, *f(x)*approaches to 6, when x approaches to 2.

Let y = f(x) be a function and *l* be a real number. Then *l* is said to be limit of function $y = f(x)$ as x approaches to a real number 'a'.

Symbolically, when $x \to a$, then $f(x) \to l$

i.e. $\lim_{x \to a} f(x) = l$

i.e. $f(x) \to l$ when $x \to a$

Example 1

Complete the following table and write the limit in notation from $y = f(x) = x^2 - 2$

x	1	2	2.2	2.4	2.6	2.8	2.9	2.99		3
y = f (x)										

Solution:

Here, $f(x) = x^2 - 2$.

Then the table can be obtained by putting the value of x as 1, 2, 2.2, … and so on simultaneously. So these values are given as follows:

x	1	2	2.2	2.4	2.6	2.8	2.9	2.99			3
$y = f(x)$	–1	2	2.84	3.76	4.76	5.84	6.41	6.94			7

Here, when x approaches to 3, the value of $f(x) = x^2 - 2$ approaches to 7.

i.e. $\lim_{x \to 3} x^2 - 2 = 7$

Example 2

Write the following table in terms of limit.

x	1	2	3	4	5	6			10
f (x) = x^3	1	8	27	64	125	216			1000

Solution:

Here x approaches to 10 from 1 and f (x) approaches to 1000 from 1.

So, symbolically we can write f (x) $\rightarrow$ 100 when $x \rightarrow 10$

i.e, when x approaches to 10 ,the value of f(x) = x^3 approaches to 1000.

i.e, in limit form , $\lim_{x \rightarrow 10} x^3 = 1000$

Exercise 2.4

1. **Find the functional value of given functions at given points.**
   a) f (x) = $3x^2 - 2x + 2$ at $x = 2$
   b) f (x) = $x^3 - x^2 - x + 1$ at $x = -2$
   c) f (x) = $2x^2 - 5x + 6$ at $x = 5$
   d) f (x) = $5x^2 + 6$ at $x = -3$
2. **Write the following statements in symbolic form.**
   (a) x approaches to 3
   b) x approaches to – 4
   c) a approaches to 10
   d) a approaches to∞
3. **Complete the following tables.**

(a)

x	1	2	3	4	5	6	7	8	9	10
f (x) =3x -4	-1	2	5							

(b)

x	3	3.1	3.2	3.3	3.6	3.8	3.9	3.99	4
$y = f(x) = x^2 - x$									

(c)

x	1	2	3	4	5	6	7	8		10
$f(x) = \frac{x}{2}$										

(d)

x	4	4.2	4.4	4.6	4.8	4.9	4.99		5
$f(x) = 3x+1$									

4. Write the following in the symbolic form of limit.

(a)

x	1	1.1	1.3	1.5	1.6	1.7	1.9	1.99	1.999	2
$f(x) = 2x$	2	2.2	2.6	3.0	3.2	3.4	3.8	3.98	3.998	4

(b) $f(x) = \frac{1}{x}$

x	1	2	3	4	5	6	7	8	9	10
$f(x) = \frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$

(c) $f(x) = \frac{1}{x^2}$

x	1	10	100	1000		∞
$f(x) = \frac{1}{x^2}$	1	0.0.1	0.0001	0.000001		0

(d) $f(x) = \frac{x+2}{2}$

x	2	4	6	8	10	12	…	120
$f(x) = \frac{x+2}{2}$	2	3	4	5	6	7	…	61

5. In an I.Q test there are two persons, one from Biology background and the next from mathematics background. They were asked to across a room with two doors of one enter and the next exit. The condition was given that "every person only can run the half the distance from their position and exit door in each step". At that time the person from Biology started to run but the person from Mathematics background sat down at the first door. Why? Discuss in small group and prepare a report.

Unit 3

Matrices

3.0 Review

In the process of the development of mathematics when matrices came into existence, they served hugely in different branches of mathematics. The very simplest but the most important application of matrices is to present data in a rectangular arrangements and use them in several decisions making.

For example, the grade sheet of a student is given as follow:

Subject	Credit hour	Total GP	Obtained GP	highest GP of the class
Nepali	5	4.0	3.5	3.8
Mathematics	5	4.0	4.0	4.0
English	5	4.0	3.8	3.9
Science	5	4.0	3.9	4.0
Social study	5	4.0	3.7	3.8
HPE	5	4.0	3.8	3.9
Opt. Mathematics	5	4.0	3.9	4.0

Here, the subjects and the data related to respected subjects are presented in the form of rows and columns. The first row gives the information of Nepali while the last row gives the information about optional mathematics. The subjects are represented by seven rows and their data are represented by four columns. Representing numbers in the form of a rectangle into rows and columns is called a Matrix. Based on the grade sheet of a student we can discuss different questions as below:

a) Which rows represents information for Social Studies?
b) Which column represents the obtained GP?
c) Which is the highest GP in Science?

Discuss in groups and present the conclusion in classroom.

Example 2

a) How many rows does the grade sheet have? What are the numbers representing in the fifth row?

b) How many columns does the grade sheet have? What are the numbers in the third columns representing?
c) What does the number in the intersection of the fifth row and the third column represent?

In mathematics we omit the columns of the subjects like Nepali, Mathematics, etc. and write only the number in the arrays of rectangle by any one of the following methods by using square brackets or round brackets.

$$M = \begin{bmatrix} 5 & 4.0 & 3.5 & 3.8 \\ 5 & 4.0 & 4.0 & 4.0 \\ 5 & 4.0 & 3.8 & 3.9 \\ 5 & 4.0 & 3.9 & 4.0 \\ 5 & 4.0 & 3.7 & 3.8 \\ 5 & 4.0 & 3.8 & 3.9 \\ 5 & 4.0 & 3.9 & 4.0 \end{bmatrix} \text{ or } M = \begin{pmatrix} 5 & 4.0 & 3.5 & 3.8 \\ 5 & 4.0 & 4.0 & 4.0 \\ 5 & 4.0 & 3.8 & 3.9 \\ 5 & 4.0 & 3.9 & 4.0 \\ 5 & 4.0 & 3.7 & 3.8 \\ 5 & 4.0 & 3.8 & 3.9 \\ 5 & 4.0 & 3.9 & 4.0 \end{pmatrix}$$

This way of writing number is called matrix. The matrix we wrote here is denoted by a capital letter M.

The rectangular arrangement of numbers in rows and columns enclosed by a pair of square or round brackets is called a matrix. The number that make the matrix are called its elements or entries.

Notation of matrix and it's order

Matrices are generally denoted by capital letters A, B, C,….. etc. and its entries by small letter like a, b, c etc. For example, $A = \begin{pmatrix} a & b & c \\ p & q & r \end{pmatrix}$ and

$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ -3 & 2 & 4 \end{pmatrix}$ are two matrices.

Now answer the following questions.

a) How many rows are there in matrix A?
b) How many columns are there in the matrix A?
c) How many rows are there in matrix B?
d) How many columns are there in matrix B?

Here, matrix A has two rows and three columns, we say that matrix A has the order of 2×3 (two by three) that doesn't mean $2 \times 3 = 6$. Likewise, matrix B has three rows and three columns; we say that matrix B has the order 3 × 3 (three by three) that doesn't mean 3 × 3 = 9.

The number of rows followed by number of columns in a matrix is called its order. If a matrix A has m rows and n columns, then it is the matrix of order m × n and written by $A_{m \times n}$.

If a matrix A has i rows and j columns, then A is of order i x j and written by $A_{i \times j}$.

For example,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} = (a_{ij}) \text{ where } i = 3 \text{ and } j = 4$$

Here, the elements a_{23} is in the second row and third column. Likewise the element in the third row and second column is a_{32}. Here a_{23} and a_{32} are different elements having different values for i and j, in which i represents the number of rows and j represents the number of columns.

Example 3

The following P, Q, R, S are four matrices.

a) $P = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ b) $Q = \begin{pmatrix} 8 \\ 9 \\ 7 \end{pmatrix}$

c) $R = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 3 & 4 & 2 \end{pmatrix}$ d) $S = \begin{pmatrix} -2 & 3 \\ 1 & -4 \\ 0 & 5 \end{pmatrix}$

In each above matrices, find

i) Number of rows and columns in each and hence the order of each matrix.
ii) Write the elements in the first row and second column of matrix P.
iii) Write the elements in the second row and third column of the matrix R.
iv) Write the element in the third row and first column of the matrix Q.

Solution:

i) a) Matrix P has one row and three columns. It has the order 1×3. Likewise, matrix Q has the order 3×1, R has the order 3×3 and S has the order 3×2.
ii) The elements in the first row and second column in matrix P is 2

$$P = \begin{pmatrix} 1 & \textcircled{2} & 3 \end{pmatrix}$$

iii) The element in the second row and third column of the matrix R is 9.

$$R = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 3 & 4 & 2 \end{pmatrix}$$

iv) The elements in the third row and first column of matrix Q is 7.

$$Q = \begin{pmatrix} 8 \\ 9 \\ 7 \end{pmatrix}$$

Example 4

If $A = \begin{pmatrix} 7 & 9 & 12 \\ 2 & 4 & 9 \\ 10 & 11 & 12 \end{pmatrix}$ be a given matrix,

a) How many elements has A?
b) What is the order of the matrix A?

Solution:

a) Matrix A has 9 elements.
b) The order of the matrix A is 3×3.

Example 5

If $P = \begin{pmatrix} 8 & 12 & 16 \\ 1 & 3 & 5 \end{pmatrix}$ write down the elements represented by a_{11}, a_{12}, a_{22}, a_{23}.

Solution:

a_{11} = An element lies in first row and first column = 8

a_{12} = An element lies in first row and second column = 12

a_{22} = An element lies in second row and second column = 3

a_{23} = An element lies in second row and third column = 5

Exercise 3.1

1. a) Define a matrix. How are matrices denoted?

   b) What is meant by the order of a matrix?

   c) How are the elements of a matrix denoted? Give an example.

   d) If $P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$, find the number of elements in P.

2. Find the order of the following matrices

a) A = $\begin{pmatrix} 11 & 12 & 13 \\ 14 & 15 & 16 \\ 17 & 18 & 19 \end{pmatrix}$ b) B = $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$

c) C = $(i \quad o \quad u)$ d) D = $\begin{pmatrix} 13 \\ 10 \\ 7 \end{pmatrix}$

3. a) In matrix A of Q. 2(a) find the elements in the second row and third column.

b) In matrix A of Q. 2(a) if the element 16 is in the second row and third column, Find the value of i and j if the elements are denoted by a_{ij}.

c) In matrix C of Q. 2(c) what is the element a_{13} equal to?

d) In matrix D of Q. 2(d), $a_{ij} = 10$, find the value of i and j.

4. If M = $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$, what is the order of matrix M? Write also the order of M in ij-form of matrix.

5. A = $\begin{pmatrix} -2 & 4 & 6 \\ 1 & 3 & -5 \\ 3 & 7 & -9 \end{pmatrix}$ is a given matrix. If its elements are written in the form a_{ij} what are the values of element a_{11}, a_{22}, a_{32}?

6. Table below gives the grade sheet of 2 students in three subjects.

Name	Maths	Science	Nepali
Kusum	3.2	3.6	3.6
Kapil	3.6	3.2	2.8

a) Construct a matrix A having elements (a_{ij}) where i = 2 and j = 3.

b) Construct a matrix B having elements (a_{ij}) where i = 2 and j = 2.

3.2 Types of Matrices

We will discuss on different types of matrices in the following:

a. Row matrix

A matrix having only one row is called a row matrix, for example

A = (0   1 -3) is a 1×3 row matrix. Therefore, a matrix a_{ij} is row matrix if i = 1.

b. Column matrix

A matrix having only one column is called a column matrix. For example $B = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ is a 3×1 column matrix. Therefore, a matrix a_{ij} is a column matrix when j = 1.

c. Square Matrix

A matrix a_{ij} is a square matrix if i = j. for example $S = \begin{pmatrix} -2 & 4 & 6 \\ 1 & 3 & -5 \\ 3 & 7 & -9 \end{pmatrix}$ is a 3×3 square matrix.

Likewise $P = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$ is a 2×2 square matrix. Did you notice here, a square matrix has equal number of rows i and columns j.

d. Rectangular matrix

A matrix in which the number of rows is not equal to the number of column is a rectangular matrix. For example,

$P = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 4 & -6 \end{pmatrix}$, $Q = \begin{pmatrix} 5 & 3 & 4 \\ 2 & 0 & -1 \\ 3 & 5 & 7 \\ -1 & 2 & 8 \end{pmatrix}$ are rectangular matrices.

e. Zero or Null matrix

A matrix is a zero matrix if each element in the matrix is zero. It is denoted by O. For example,

$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, is null or zero matrix of order 3× 3.

f. Diagonal Matrix

A square matrix in which the main diagonal element (from top left to bottom right) are non-zero and all the elements except main diagonal are zeros is a diagonal matrix. For example,

$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ etc. are diagonal matrices.

g. Scalar Matrix

It is a diagonal matrix in which all elements in the main diagonal are equal (or the same). For example,

$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ etc. are the scalar matrices.

h. Unit or Identity Matrix

It is a scalar matrix having each element in the main diagonal equal to 1. For example,

$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ etc. are unit or identity matrices.

i. Triangular Matrix

A square matrix A is upper triangular matrix if elements below main diagonal are all zeros. For example,

$M = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, $N = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}$ etc. are upper triangular matrix.

A square matrix B is a lower triangular matrix if elements above main diagonal are all zeros. For example:

$P = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 2 & 4 \end{pmatrix}$ etc. are lower triangular matrix.

j. Symmetric Matrix

It is a square matrix that doesn't change if its row and columns are interchanged. In general, a matrix A is symmetric if $a_{ij} = a_{ji}$. For example,

$A = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 1 \end{pmatrix}$ etc. are symmetric matrices.

k. Equal Matrices

Two matrices of the same order are equal if and only if their corresponding elements are equal. For example, If $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & -3 \\ -2 & 4 & 7 \end{pmatrix}$ then A = B if and only if a = 1, b = 2, c = –3, d = –2, e = 4 and f = 7.

Example 1

a) Find a, b, c, d if $\begin{pmatrix} a & 2 \\ 4 & b \end{pmatrix} = \begin{pmatrix} 3 & c \\ d & 5 \end{pmatrix}$

b) Find the value of x and y if $\begin{pmatrix} x+y & 7 \\ 5 & x-y \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 5 & 2 \end{pmatrix}$

Solution:

a) Here $\begin{pmatrix} a & 2 \\ 4 & b \end{pmatrix} = \begin{pmatrix} 3 & c \\ d & 5 \end{pmatrix}$.

Equating corresponding elements in two equal matrices, we get

a = 3, b = 5, c = 2 and d = 4 as required.

b) Here, $\begin{pmatrix} x+y & 7 \\ 5 & x-y \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 5 & 2 \end{pmatrix}$

By equating corresponding elements in two equal matrices, we get

$x + y = 6$…………i)

$x - y = 2$…………ii)

Adding (i) and (ii) we get

$2x = 8$ this implies $x = 4$

From equation (i)

$4 + y = 6$

Or, y = 2.

Hence, $x = 4$, $y = 2$ are required values.

Example 2

If matrix $A = (a_{ij})_{2\times3}$ and $a_{ij} = (i\times j)^2$, construct a matrix A.

Solution:

We have $A = (a_{ij})_{2\times3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$, then

$a_{11} = (1\times1)^2 = 1^2 = 1$ $a_{12} = (1\times2)^2 = 2^2 = 4$ $a_{13} = (1\times3)^2 = 3^2 = 9$

$a_{21} = (2\times1)^2 = 2^2 = 4$ $a_{22} = (2\times2)^2 = 4^2 = 16$ $a_{23} = (2\times3)^2 = 6^2 = 36$

Therefore, A = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

i.e. A = $\begin{pmatrix} 1 & 4 & 9 \\ 4 & 16 & 36 \end{pmatrix}$ is required matrix.

Example 3

If $\begin{pmatrix} x-1 & 2q-4 \\ 3p-6 & y+2 \end{pmatrix}$ is an identity matrix, find the value of *x*, y , p and q.

Solution:

Here, $\begin{pmatrix} x-1 & 2q-4 \\ 3p-6 & y+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, By equality of matrices, we get.

$x - 1 = 1$ this implies $x = 2$

$y + 2 = 1$ this implies $y = -1$

$3p - 6 = 0$ this implies $p = 2$

$2q - 4 = 0$ this implies $q = 2$; are required values.

Exercise: 3.2

1. **Define the following matrices with example.**

   a) Row matrix b) Column matrix

   c) Square matrix d) Diagonal matrix

   e) Scalar matrix f) Triangular matrix

   f) Symmetric matrix g) Identity matrix

2. **State the types of following matrices.**

   a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 11 \\ 13 \\ 17 \end{pmatrix}$ c) $\begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{pmatrix}$

   d) $\begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix}$ e) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{pmatrix}$

3. a. Construct a square matrix $M_{2 \times 2}$, if M = (a_{ij}) and $a_{ij} = 2i + j$.

   b. Construct a matrix $N_{2 \times 3}$ if N = (a_{ij}) and $a_{ij} = i - j$

   c. A = (a_{ij}) is a given matrix where $a_{ij} = (i \times j)^2$, construct a square matrix $A_{3 \times 3}$.

d. If $P = (a_{ij})$ is a given matrix where $a_{ij} = (i - j)^2$, construct a square matrix $P_{3 \times 3}$.

4. (a) Find a, b, c, d, if $\begin{pmatrix} a & 1 \\ 3 & b \end{pmatrix} = \begin{pmatrix} 2 & c \\ d & -2 \end{pmatrix}$

(b) If $\begin{pmatrix} p + q \\ p - q \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$, find the value of p and q.

(c) If A = $\begin{pmatrix} x - 1 & 3 \\ 5 & y \end{pmatrix}$, B = $\begin{pmatrix} -3 & x + z \\ w - y & 2 \end{pmatrix}$ and A = B find the value of w, x, y, and z.

(d) If $\begin{pmatrix} x - 6 & 4y - 6 \\ 5p - 10 & q + 2 \end{pmatrix}$ is an identity matrix, find the value of x, y, p and q.

3.3 Operation on Matrices

In this chapter we shall discuss on the addition, subtraction and multiplication operations with matrices.

Look at the following example:

Example 1

a. Addition of Matrices

Nepal Airlines have two kinds of flight services; namely domestic services and international services.

During the first three days of the last week, the flight services of Nepal Airlines were recorded as in the following:

Lot A

	Domestic service	International service
Sunday	14	6
Monday	30	4
Tuesday	36	5

During the last three days of the last week the flight services of Nepal Airlines were recorded as:

Lot B

	Domestic Service	International Service
Wednesday	40	5
Thursday	42	7
Friday	38	6

How many flights were made in these two categories in the last week?

Solution:

To find the total flight service in these two lots in two categories, we add the flights in corresponding rows as in the following.

Lot A + Lot B

Days	National flight	International flight
Sunday + Wednesday	14 + 40 = 54	6 + 5 = 11
Monday + Thursday	30 + 42 = 72	4 + 7 = 11
Tuesday + Friday	36 + 38 = 74	5 + 6 = 11

Representing the above information in matrix from we have.

$$A = \begin{pmatrix} 14 & 6 \\ 30 & 4 \\ 36 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 40 & 5 \\ 42 & 7 \\ 38 & 6 \end{pmatrix}$$

Adding the matrices, A and B by adding corresponding entries, we get.

$$\begin{aligned} A+B &= \begin{pmatrix} 14 & 6 \\ 30 & 4 \\ 36 & 5 \end{pmatrix} + \begin{pmatrix} 40 & 5 \\ 42 & 7 \\ 38 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 14+40 & 6+5 \\ 30+42 & 4+7 \\ 36+38 & 5+6 \end{pmatrix} \\ &= \begin{pmatrix} 54 & 11 \\ 72 & 11 \\ 74 & 11 \end{pmatrix} \end{aligned}$$

Here, both added matrices have the same order, and their sum is also of the same order.

Two matrices of the same order $A_{m \times n}$ and $B_{m \times n}$ could be added by adding corresponding entries and the sum (A+B) will have the same order as A or B; i.e. $A_{m \times n} + B_{m \times n} = (A + B)_{m \times n}$.

By definition, we have.

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_{2\times 3} + \begin{pmatrix} p & q & r \\ s & t & u \end{pmatrix}_{2\times 3} = \begin{pmatrix} a+p & b+q & c+r \\ d+s & e+t & f+u \end{pmatrix}_{2\times 3}$$

Example 2

If $M = \begin{pmatrix} 3 & 7 & 9 \\ 11 & 13 & 15 \end{pmatrix}$ and $N = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$ find M + N and N + M.

$$M + N = \begin{pmatrix} 3 & 7 & 9 \\ 11 & 13 & 15 \end{pmatrix} + \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 3+2 & 7+4 & 9+6 \\ 11+8 & 13+10 & 15+12 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 11 & 15 \\ 19 & 23 & 27 \end{pmatrix}$$

$$N + M = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix} + \begin{pmatrix} 3 & 7 & 9 \\ 11 & 13 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} 2+3 & 4+7 & 6+9 \\ 8+11 & 10+13 & 12+15 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 11 & 15 \\ 19 & 23 & 27 \end{pmatrix} = M + N$$

Did you notice here M + N = N + M. This property of addition is called commutative property of addition.

While adding matrices the order at which they are added does not matter. This means, commutative property holds in matrix addition.

b. Subtraction of matrices:

Subtraction of two matrices of same order is also carried out exactly in the same way as adding matrices. In subtraction we subtract the corresponding elements.

Example 3

If $A = \begin{pmatrix} 5 & 7 & 3 \\ 4 & 6 & -5 \\ -7 & 3 & 10 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 & -1 \\ -3 & 4 & 2 \\ 5 & 1 & -8 \end{pmatrix}$, find A-B and B-A

Solution:

$$A - B = \begin{pmatrix} 5 & 7 & 3 \\ 4 & 6 & -5 \\ -7 & 3 & 10 \end{pmatrix} - \begin{pmatrix} 3 & 4 & -1 \\ -3 & 4 & 2 \\ 5 & 1 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} 5-3 & 7-4 & 3+1 \\ 4+3 & 6-4 & -5-2 \\ -7-5 & 3-1 & 10+8 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 4 \\ 7 & 2 & -7 \\ -12 & 2 & 18 \end{pmatrix}$$

$$B - A = \begin{pmatrix} 3 & 4 & -1 \\ -3 & 4 & 2 \\ 5 & 1 & -8 \end{pmatrix} - \begin{pmatrix} 5 & 7 & 3 \\ 4 & 6 & -5 \\ -7 & 3 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 3-5 & 4-7 & -1-3 \\ -3-4 & 4-6 & 2+5 \\ 5+7 & 1-3 & -8-10 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -3 & -4 \\ -7 & -2 & 7 \\ 12 & -2 & -18 \end{pmatrix}$$

Here $A - B \neq B - A$.

This means subtraction of matrices does not obey commutative rule.

c. Properties of Matrix Addition

i. Closure property

Consider $P = \begin{pmatrix} 11 & 13 \\ 17 & 19 \end{pmatrix}_{2\times 2}$ and $Q = \begin{pmatrix} 8 & 10 \\ 12 & 14 \end{pmatrix}_{2\times 2}$

Then $P + Q = \begin{pmatrix} 11 & 13 \\ 17 & 19 \end{pmatrix}_{2\times 2} + \begin{pmatrix} 8 & 10 \\ 12 & 14 \end{pmatrix}_{2\times 2}$

$$= \begin{pmatrix} 11+8 & 13+10 \\ 17+12 & 19+14 \end{pmatrix}_{2\times 2}$$

$$= \begin{pmatrix} 19 & 23 \\ 29 & 33 \end{pmatrix}_{2\times 2}$$

> The sum of two matrices of the same order gives the matrix of the same order. This property of matrix addition is called the Closure Property of matrix addition.

ii. Commutative Property

Consider the matrices $A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 5 \\ -3 & 7 \end{pmatrix}$.

Now,

A + B $= \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} + \begin{pmatrix} -1 & 5 \\ -3 & 7 \end{pmatrix}$

$= \begin{pmatrix} 2-1 & 4+5 \\ 6-3 & 8+7 \end{pmatrix} = \begin{pmatrix} 1 & 9 \\ 3 & 15 \end{pmatrix}$

Again,

B + A $= \begin{pmatrix} -1 & 5 \\ -3 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$

$= \begin{pmatrix} -1+2 & 5+4 \\ -3+6 & 7+8 \end{pmatrix} = \begin{pmatrix} 1 & 9 \\ 3 & 15 \end{pmatrix}$

Here, A + B = B + A

Matrix addition follows commutative property.

iii. Associative Property

Let A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, B = $\begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix}$ and C = $\begin{pmatrix} -3 & 4 \\ 5 & -6 \end{pmatrix}$ be the given matrices of the same order.

Now (A+B) = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 9 \end{pmatrix}$

And (A+B)+C = $\begin{pmatrix} 3 & -1 \\ -1 & 9 \end{pmatrix} + \begin{pmatrix} -3 & 4 \\ 5 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 4 & 3 \end{pmatrix}$……….(i)

Again, (B+C) = $\begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix} + \begin{pmatrix} -3 & 4 \\ 5 & -6 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

And, A+(B+C) = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 4 & 3 \end{pmatrix}$………….(ii)

From (i) and (ii) (A+B)+C = A+(B+C). This property of addition of matrices is called the Associative Property of Addition.

To add matrices with more than two addends, they can be added by desired grouping. This property of matrix addition is called the Associative Property of Addition.

iv. Identity Property:

Let A = $\begin{pmatrix} 12 & 14 & 16 \\ 9 & 12 & 15 \end{pmatrix}$ and Z = $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then A+Z $= \begin{pmatrix} 12 & 14 & 16 \\ 9 & 12 & 15 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 12 & 14 & 16 \\ 9 & 12 & 15 \end{pmatrix} = A$

And, Z+A $= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 12 & 14 & 16 \\ 9 & 12 & 15 \end{pmatrix}$

$= \begin{pmatrix} 12 & 14 & 16 \\ 9 & 12 & 15 \end{pmatrix} = A$

Here, Z + A = A + Z = A.

If Z is the zero matrix of same order as A such that A+Z = Z+A = A, then, Z is called the additive identity of matrix A.

v. Additive Inverse Law of matrix

Let $A = \begin{pmatrix} 1 & 4 \\ 9 & 16 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -4 \\ -9 & -16 \end{pmatrix}$

Then A+B $= \begin{pmatrix} 1 & 4 \\ 9 & 16 \end{pmatrix} + \begin{pmatrix} -1 & -4 \\ -9 & -16 \end{pmatrix}$

$= \begin{pmatrix} -1+1 & -4+4 \\ -9+9 & -16+16 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$……(i)

And B+A $= \begin{pmatrix} -1 & -4 \\ -9 & -16 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 9 & 16 \end{pmatrix}$

$= \begin{pmatrix} -1+1 & -4+4 \\ -9+9 & -16+16 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$……(ii)

From (i) and (ii) A+B = B+A = 0.

Here B is additive inverse of A and A is additive inverse of B.

Two matrices A and B are such that A+B = B+A = O then A and B are additive inverse of each other.

When A+B = O, then A = -B and A+B = -B+B = O is the application of additive inverse property of matrix addition.

Example 4

If $A = \begin{pmatrix} 8 & -6 \\ 3 & 0 \end{pmatrix}$ and A+B = O find the matrix B, where O is 2×2 null matrix

Solution:

We have A + B = O

or, B = O - A

or, B = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$-$\begin{pmatrix} 8 & -6 \\ 3 & 0 \end{pmatrix}$

or, B = $\begin{pmatrix} 0-8 & 0+6 \\ 0-3 & 0-0 \end{pmatrix}$

or, B = $\begin{pmatrix} -8 & 6 \\ -3 & 0 \end{pmatrix}$ as required

Example 5

Let A = $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$, B = $\begin{pmatrix} 5 & -2 \\ 0 & 7 \end{pmatrix}$ and C = $\begin{pmatrix} -2 & 8 \\ 6 & -3 \end{pmatrix}$, then prove that

(a) A + B = B + A

(b) (A + B) + C = A + (B + C)

(c) (A - C) ≠ (C - A)

Solution:

a. Here, A + B = $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$ + $\begin{pmatrix} 5 & -2 \\ 0 & 7 \end{pmatrix}$

= $\begin{pmatrix} 2+5 & 1-2 \\ 3+0 & 0+7 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ 3 & 7 \end{pmatrix}$ …………..… (i)

B + A = $\begin{pmatrix} 5 & -2 \\ 0 & 7 \end{pmatrix}$ + $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$

= $\begin{pmatrix} 5+2 & -2+1 \\ 0+3 & 7+0 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ 3 & 7 \end{pmatrix}$ ………………. (ii)

Here from (i) and (ii) A + B = B + A

b) We have, (A + B) = $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$ + $\begin{pmatrix} 5 & -2 \\ 0 & 7 \end{pmatrix}$

= $\begin{pmatrix} 2+5 & 1-2 \\ 3+0 & 0+7 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ 3 & 7 \end{pmatrix}$

(A+B)+C = $\begin{pmatrix} 7 & -1 \\ 3 & 7 \end{pmatrix}$ + $\begin{pmatrix} -2 & 8 \\ 6 & -3 \end{pmatrix}$

= $\begin{pmatrix} 7-2 & -1+8 \\ 3+6 & 7-3 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 9 & 4 \end{pmatrix}$ ……………. (i)

Again, (B+C) = $\begin{pmatrix} 5 & -2 \\ 0 & 7 \end{pmatrix}$+$\begin{pmatrix} -2 & 8 \\ 6 & -3 \end{pmatrix} = \begin{pmatrix} 5-2 & -2+8 \\ 0+6 & 7-3 \end{pmatrix}$

∴ (B+C) = $\begin{pmatrix} 3 & 6 \\ 6 & 4 \end{pmatrix}$

And A+(B+C) = $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$+$\begin{pmatrix} 3 & 6 \\ 6 & 4 \end{pmatrix}$

$$\begin{pmatrix} 2+3 & 1+6 \\ 3+6 & 0+4 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 9 & 4 \end{pmatrix} \text{..........(ii)}$$

From (i) and (ii) (A+B)+C = A+(B+C)

c) (A-C) = $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} -2 & 8 \\ 6 & -3 \end{pmatrix}$

$= \begin{pmatrix} 2-2 & 1-8 \\ 3-6 & 0+3 \end{pmatrix}$

$= \begin{pmatrix} 0 & -7 \\ -3 & 3 \end{pmatrix}$............................(i)

And (C-A) = $\begin{pmatrix} -2 & 8 \\ 6 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$

$= \begin{pmatrix} -2+2 & 8-1 \\ 6-3 & -3+0 \end{pmatrix} = \begin{pmatrix} 0 & 7 \\ 3 & -3 \end{pmatrix}$ ....(ii)

Hence, from (i) and (ii) A-C ≠ C-A

Exercise 3.3

1. (a) Under what conditions two matrices could be added?
   (b) How do you define addition of two matrices?
   (c) How do you define subtraction of two matrices?
   (d) List down five properties of matrix addition.
2. From the matrices given below make as many pair of matrices as you can that could be added or subtracted each other.

   M $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ N = $\begin{pmatrix} 3 & 4 \\ 7 & 9 \end{pmatrix}$ P = $\begin{pmatrix} 11 \\ 13 \\ 14 \end{pmatrix}$

   Q = $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \end{pmatrix}$ R = $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$ S = $(7 \quad 8 \quad 9)$

   T = $\begin{pmatrix} e & -f & g \\ -h & i & -j \end{pmatrix}$
3. (a) Using the matrices given in question 2, carry out the following operations.

   (i) Q+R (iii) M-N (v) T+(Q+R) (vii) (T+Q)-R

   (ii) R+Q (iv) N-M (vi) (T+Q)+R (viii) T+(Q-R)

   (b) Find the additive inverse of the matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

4 (a) If $\begin{pmatrix}5 & 6\\7 & x\end{pmatrix}+\begin{pmatrix}1 & y\\0 & 1\end{pmatrix} = \begin{pmatrix}6 & 7\\7 & 3\end{pmatrix}$, find the value of x and y.

(b) If $\begin{pmatrix}9 & x\\10 & 7\end{pmatrix}-\begin{pmatrix}5 & 2\\7 & 1\end{pmatrix} = \begin{pmatrix}4 & 3\\3 & 4\end{pmatrix}$, find the values of x and y

(c) If $\begin{pmatrix}3x-2 & 5y+4\\2 & 4+2x\end{pmatrix} = \begin{pmatrix}x+2 & y-4\\2 & z-2\end{pmatrix}$, find x, y and z.

(d) If $\begin{pmatrix}x-1 & -4\\y+3 & 5\end{pmatrix}$ and $\begin{pmatrix}-1 & 4\\-4 & -5\end{pmatrix}$ are additive inverse to each other, find the value of x and y.

5. (a) If A = $\begin{pmatrix}4 & 5 & 6\end{pmatrix}$, B = $\begin{pmatrix}-3 & 7 & 2\end{pmatrix}$ and C = $\begin{pmatrix}8 & 9 & -4\end{pmatrix}$, find the matrix X such that

(i) X = A+B+C (ii) A-X = B+C (iii) X-C = B

(b) If A = $\begin{pmatrix}2\\3\end{pmatrix}$, B = $\begin{pmatrix}-2\\1\end{pmatrix}$ and C = $\begin{pmatrix}1\\-3\end{pmatrix}$, find the matrix Y such that

(i) Y = A+B-C (ii) Y-A = B (iii) A+Y = B+C

6. Solve the following equation for the matrix X

(a) X+$\begin{pmatrix}2 & 1\\3 & -2\end{pmatrix} = \begin{pmatrix}5 & -1\\-2 & 3\end{pmatrix}$

(b) X-$\begin{pmatrix}5 & 2\\-2 & 1\end{pmatrix} = \begin{pmatrix}7 & -3\\-4 & 8\end{pmatrix}$

(c) X+$\begin{pmatrix}2 & 4\end{pmatrix} = \begin{pmatrix}6 & 8\end{pmatrix}$

(d) X-$\begin{pmatrix}5 & 3\end{pmatrix}$ = 2X +$\begin{pmatrix}7 & 8\end{pmatrix}$

(e) X+$\begin{pmatrix}5\\2\\7\end{pmatrix} = \begin{pmatrix}9\\-3\\4\end{pmatrix}$

(f) X-$\begin{pmatrix}8\\-2\\4\end{pmatrix} = \begin{pmatrix}-3\\2\\6\end{pmatrix}$

7. If A = $\begin{pmatrix}5 & 8\\-9 & 3\end{pmatrix}$, find

a) a matrix B such that A+B = O

b) a matrix C such that A+C = A

c) a matrix D such that A+D = B+C.

8. If A = $\begin{pmatrix}2 & 1\\3 & -5\end{pmatrix}$, B = $\begin{pmatrix}5 & 2\\-3 & 4\end{pmatrix}$ and C = $\begin{pmatrix}-8 & 2\\7 & 6\end{pmatrix}$, show that.

(a) A+B = B+A

(b) (A+B)+C = A+(B+C)

(c) A-B-C = A-(B+C)

(d) A+(-A) = O

9. Tables below show the order of sports T-shirts made by a sporting house. Using matrices, find the total order made in these two lots.

Lot A			
	Small	Medium	Large
Red	48	62	91
Blue	38	57	55
Green	62	98	37
Yellow	55	65	75

Lot B			
	Small	Medium	Large
Red	58	72	80
Blue	42	67	70
Green	50	80	65
Yellow	45	55	85

3.4 Transpose of a Matrix

Consider the matrix A = $\begin{pmatrix} 2 & 3 & 4 \\ 5 & -2 & 7 \end{pmatrix}$, and answer the following questions.

a. What is the order of matrix A?

b. If you interchange the row and columns of the matrix A, what matrix will you get? Name this matrix by A^T

c. What is the order of the matrix A^T?

> A matrix obtained by interchanging the rows and columns is called the transpose of the given matrix. If the given matrix is A, its transpose is denoted by A^T. Symbolically if A_{ij} is a matrix then A^t_{ji} is transpose of A.

Note that if a matrix A has order m×n, then its transpose A^T will have order n×m.

Example 1

Find the transpose of the following matrices:

a) A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ b) I = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) $S = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -7 \\ 3 & -7 & 5 \end{pmatrix}$ d) $R = \begin{pmatrix} 5 & 9 & 7 \end{pmatrix}$

Solution:

a) Here $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times2}$

Then $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}_{2\times2}$

Did you notice here, the order of A and A^T is equal !

b) Here, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3}$

Then $I^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3}$

Did you notice here, $I = I^T$!

c) Here $S = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -7 \\ 3 & -7 & 5 \end{pmatrix}_{3\times3}$

Then $S^T = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -7 \\ 3 & -7 & 5 \end{pmatrix}_{3\times3}$

Did you notice here, $S = S^T$

d) Here, $R = \begin{pmatrix} 5 & 9 & 7 \end{pmatrix}_{1\times3}$

Then $R^T = \begin{pmatrix} 5 \\ 9 \\ 7 \end{pmatrix}_{3\times1}$

Form above examples, we can observe:

1. The Transpose of a square matrix has the same order as the given matrix.
2. The transpose of an identity matrix is the matrix itself.
3. The transpose of a symmetric matrix is the matrix itself.
4. The transpose of a row matrix is the column matrix of different order.

Properties of Transpose Matrices

1. Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}_{2\times2}$

And $(A^T)^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

The transpose of the transpose of a given matrix is the matrix itself.

2. Let $M = \begin{pmatrix} 1 & 12 \\ 14 & 3 \end{pmatrix}$ and $N = \begin{pmatrix} 4 & 9 \\ 15 & 21 \end{pmatrix}$

Now, $M+N = \begin{pmatrix} 1 & 12 \\ 14 & 3 \end{pmatrix}+\begin{pmatrix} 4 & 9 \\ 15 & 21 \end{pmatrix} = \begin{pmatrix} 5 & 21 \\ 29 & 24 \end{pmatrix}$

And $(M+N)^T = \begin{pmatrix} 5 & 29 \\ 21 & 24 \end{pmatrix}$……………..(i)

Again, $M^T = \begin{pmatrix} 1 & 14 \\ 12 & 3 \end{pmatrix}$ and $N^T = \begin{pmatrix} 4 & 15 \\ 9 & 21 \end{pmatrix}$

Then $M^T+N^T = \begin{pmatrix} 1 & 14 \\ 12 & 3 \end{pmatrix}+\begin{pmatrix} 4 & 15 \\ 9 & 21 \end{pmatrix} = \begin{pmatrix} 5 & 29 \\ 21 & 24 \end{pmatrix}$………(ii)

From equation (i) and (ii) $(M+N)^T = M^T+N^T$.

The transpose of sum of two or more matrix is equal to the sum of their transposes.

3. Let $P = \begin{bmatrix} 40 & 42 \\ 41 & 43 \end{bmatrix}$ and k be any scalar,

Then $kP = k\begin{bmatrix} 40 & 42 \\ 41 & 43 \end{bmatrix} = \begin{bmatrix} 40k & 42k \\ 41k & 43k \end{bmatrix}$

And $(kP)^T = \begin{bmatrix} 40k & 42k \\ 41k & 43k \end{bmatrix} = k\begin{bmatrix} 40 & 41 \\ 42 & 43 \end{bmatrix} = kP^T$

If P is any matrix and k is a scalar then $(kP)^T = kP^T$

Exercise 3.4

1. a) What is meant by Transpose of matrix? Give an example.

   b) List down the three properties of Transpose of a matrix.

2. Find the transpose of the following matrices.

   a) $A = \begin{pmatrix} p & q & r \end{pmatrix}$ b) $B = \begin{pmatrix} m \\ n \\ p \end{pmatrix}$

   c) $C = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ -2 & 5 \end{pmatrix}$ d) $D = \begin{pmatrix} 3 & 5 & -9 \\ -2 & -7 & 4 \end{pmatrix}$

e) $E = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

3. If $M = \begin{pmatrix} 1 & 5 \\ 2 & 7 \end{pmatrix}$ and $N = \begin{pmatrix} -3 & 6 \\ 2 & 8 \end{pmatrix}$, prove that.

a) The order of M and M^T is the same.

b) The order of N and N^T is the same.

c) $(M^T)^T = M$

d) $(N^T)^T = N$

e) $(M+N)^T = M^T+N^T$

4. If $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, prove that

a) $A^T = A$ b) $B^T = B$.

c) Write your finding in a sentence.

5. If $P = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -1 & 4 \\ -1 & 2 & 1 \\ 4 & 1 & 3 \end{pmatrix}$ verify that,

a) $P^T = P$ b) $Q^T = Q$.

c) Write your findings in words.

6. If $P = \begin{pmatrix} 2 & 3 & -4 \\ 3 & 5 & -9 \\ -4 & -9 & 8 \end{pmatrix}$ show that

a) $P^T = P$. b) Write your findings in words.

7. If $R = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 6 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 4 & 6 \end{pmatrix}$

a) Show that $R^T = S$ and $S^T = R$.

b) Write your findings in words.

8) If $Q = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ show that $Q+Q^T$ is a symmetric matrix. Can this be generalized to any square matrix?

3.5 Multiplication of Matrices

a) **Multiplication of a matrix by a scalar**

Consider a matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then A+A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$+$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Or, 2A = $\begin{pmatrix} a+a & b+b \\ c+c & d+d \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix}$

Again A+A+A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Or, 3A = $\begin{pmatrix} a+a+a & b+b+b \\ c+c+c & d+d+d \end{pmatrix} = \begin{pmatrix} 3a & 3b \\ 3c & 3d \end{pmatrix}$

Here, to the matrix A, 2A is the scalar multiplication of A by 2 and 3A is the scalar multiplication of A by 3. We have seen here when a matrix A is multiplied by 2, each of its elements are multiplied by 2 and when A is multiplied by 3, each of its elements are multiplied by 3 and so on.

If A is any matrix and k is a scalar, then kA is a matrix obtained by multiplying each element of A by k.

Example 1

If A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ find kA and hence 2A, 3A, 4A.

Solution:

We have A = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Then kA = k$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1k & 2k & 3k \\ 4k & 5k & 6k \end{pmatrix}$

And, 2A = 2$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$

3A = 3$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$

Example 2

M = $\begin{pmatrix} 9 & 1 \\ 5 & 3 \end{pmatrix}$ and N = $\begin{pmatrix} 1 & 5 \\ 7 & -11 \end{pmatrix}$ and P is a 2 × 2 square matrix and if 3M + 5N + 2P = O, where O is a zero matrix of order 2 × 2, find the matrix P.

Solution:

Here, 3M + 5N + 2P = 0, given.

or, $3\begin{pmatrix} 9 & 1 \\ 5 & 3 \end{pmatrix} + 5\begin{pmatrix} 1 & 5 \\ 7 & -11 \end{pmatrix} + 2P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

or, $\begin{pmatrix} 27 & 3 \\ 15 & 9 \end{pmatrix} + \begin{pmatrix} 5 & 25 \\ 35 & -55 \end{pmatrix} + 2P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

or, $\begin{pmatrix} 32 & 28 \\ 50 & -46 \end{pmatrix} + 2P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

or, $2P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 32 & 28 \\ 50 & -48 \end{pmatrix}$

or, $2P = \begin{pmatrix} -32 & -28 \\ -50 & 48 \end{pmatrix}$

or, $P = \frac{1}{2}\begin{pmatrix} -32 & -28 \\ -50 & 48 \end{pmatrix}$

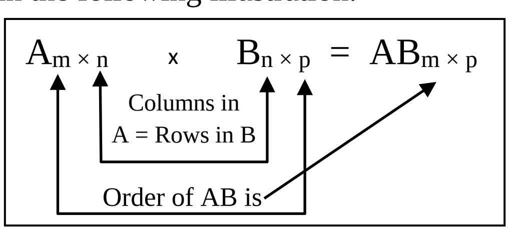
or, $P = \begin{pmatrix} -16 & -14 \\ -25 & 24 \end{pmatrix}$

b) Multiplication of matrices

The multiplication of a matrix by another matrix is defined by the rule "row versus column" as in the following:

$$(a \quad b \quad c)_{1\times 3} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{3\times 1} = (ax + \quad by + \quad cz)_{1\times 1}$$

Here, the column of the first matrix (multiplier) is equal to the row of the second matrix, (the multiplicand) that provides a check for the existence (confirmability) of the multiplication of two matrices, that is stated schematically in the following illustration.



There are many examples of matrix multiplication in day to day activities. Here is one for example. Tables below show the weekly supply of commodities in a hostel and their corresponding prices.

Table 1

Weekly purchase	Rice	Pulses
First week	500kg	30kg
Second week	600kg	32kg
Third week	550kg	25kg

Table 2

Commodities	Price
Rice	80/kg
Pulses	120/kg

To work out the total cost for the last three weeks, we work out it as in the following:

Cost of commodities for the first week

$500 \times 80 + 30 \times 120 = 43600$

This can be done by writing the rows and columns of the matrix for the first week and apply "row versus column" as in the following.

$(500 \quad 32)_{1\times2}\begin{pmatrix}80\\120\end{pmatrix}_{2\times1} = (500 \times 80 + 30 \times 120) = (43600)_{1\times1}$

Similarly, for the second and third weeks the prices are

$(600 \quad 32)_{1\times2}\begin{pmatrix}80\\120\end{pmatrix}_{2\times1} = (600 \times 80 + 32 \times 120) = (51840)_{1\times1}$

And, $(550 \quad 25)\begin{pmatrix}80\\120\end{pmatrix} = (550 \times 80 + 25 \times 120) = (45800)_{1\times1}$

The process might be worked out in compact form as:

$$\begin{pmatrix}500 & 30\\600 & 32\\550 & 25\end{pmatrix}\begin{pmatrix}80\\120\end{pmatrix} = \begin{pmatrix}500\times80+30\times120\\600\times80+32\times120\\550\times80+25\times120\end{pmatrix} = \begin{pmatrix}43600\\51840\\45800\end{pmatrix}$$

Two matrices $A_{m\times n}$ and $B_{p\times q}$ are said to be confirmable for matrix multiplication if and only if (iff) n = p and the product matrix will have the order m × q.

Multiplication of two matrices could be explained as in the following.

Consider $A = \begin{pmatrix}a & b\\c & d\end{pmatrix}$ and $B = \begin{pmatrix}w & x\\y & z\end{pmatrix}$

Then, $AB = \begin{pmatrix}\boxed{a \quad b}\\c \quad d\end{pmatrix}\begin{pmatrix}\boxed{\begin{matrix}w\\y\end{matrix}} & \begin{matrix}x\\z\end{matrix}\end{pmatrix} = \begin{pmatrix}aw+by & \dots\dots\\\dots\dots & \dots\dots\end{pmatrix}$ step-1: row1 × column1

$AB = \begin{pmatrix}\boxed{a \quad b}\\c \quad d\end{pmatrix}\begin{pmatrix}\begin{matrix}w\\y\end{matrix} & \boxed{\begin{matrix}x\\z\end{matrix}}\end{pmatrix} = \begin{pmatrix}aw+by & ax+bz\\\dots\dots & \dots\dots\end{pmatrix}$ step-2: row1 × column2

$B = \begin{pmatrix} a & b \\ \boxed{c \quad d} \end{pmatrix} \begin{pmatrix} \boxed{w} & x \\ \boxed{y} & z \end{pmatrix} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & \ldots\ldots\ldots \end{pmatrix}$ step-3: row2 × column1

$AB = \begin{pmatrix} a & b \\ \boxed{c \quad d} \end{pmatrix} \begin{pmatrix} w & \boxed{x} \\ y & \boxed{z} \end{pmatrix} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dx \end{pmatrix}$ step-4: row2×column2

Example 3

Given that $A = \begin{pmatrix} 5 & 8 \\ 2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & -4 \\ 3 & 2 \end{pmatrix}$, find

a) A×A b) A×B c) B×A.

Solution:

a) Here, A×A $= \begin{pmatrix} 5 & 8 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 5 & 8 \\ 2 & -3 \end{pmatrix}$

$= \begin{pmatrix} 5 \times 5 + 8 \times 2 & 5 \times 8 + 8 \times (-3) \\ 2 \times 5 + (-3) \times 2 & 2 \times 8 + (-3) \times (-3) \end{pmatrix}$

$= \begin{pmatrix} 41 & 16 \\ 4 & 25 \end{pmatrix}$

b) Here, A×B $= \begin{pmatrix} 5 & 8 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 7 & -4 \\ 3 & 2 \end{pmatrix}$

$= \begin{pmatrix} 5 \times 7 + 8 \times 3 & 5 \times (-4) + 8 \times 2 \\ 2 \times 7 + (-3) \times 3 & 2 \times (-4) + (-3) \times 2 \end{pmatrix}$

$= \begin{pmatrix} 59 & -4 \\ 15 & -14 \end{pmatrix}$

c) Here, B×A $= \begin{pmatrix} 7 & -4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 5 & 8 \\ 2 & -3 \end{pmatrix}$

$= \begin{pmatrix} 7 \times 5 + (-4) \times 2 & 7 \times 8 + (-4) \times (-3) \\ 3 \times 5 + 2 \times -3 & 3 \times 8 + 2 \times -3 \end{pmatrix}$

$= \begin{pmatrix} 27 & 68 \\ 9 & 18 \end{pmatrix}$

From (b) and (c) AB≠BA.

Matrix multiplication is not commutative.

Example 4

If $A = \begin{pmatrix} 2 & 0 \\ -3 & 4 \\ 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$ find AB and BA where ever applicable.

Solution:

Here A = $\begin{pmatrix} 2 & 0 \\ -3 & 4 \\ 5 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$

Since the column of matrix A equals to the rows of matrix B, so, A × B exists.

Therefore, A × B $= \begin{pmatrix} 2 & 0 \\ -3 & 4 \\ 5 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 2 \times 1 + 0 \times (-2) & 2 \times 5 + 0 \times 3 \\ (-3) \times 1 + 4 \times (-2) & (-3) \times 5 + 4 \times 3 \\ 5 \times 1 + 2(-2) & 5 \times 5 + 2 \times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 10 \\ -11 & -3 \\ 1 & 31 \end{pmatrix}$$

Again B = $\begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$ and A = $\begin{pmatrix} 2 & 0 \\ -3 & 4 \\ 5 & 2 \end{pmatrix}$

Here, the number of column of first matrix B is not equal to the number of rows of A, and hence by the conformability of matrix multiplication B×A does not exist.

Example 5

Find the matrix X if $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$X = $\begin{pmatrix} 20 & 12 \\ 5 & 3 \end{pmatrix}$

Solution: Here,

The order of the multiplier matrix $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is 2×1 and the order of the product matrix is 2×2, the order of the multiplication matrix X must be 1×2.

Let $(a \quad b)$be required matrix.

Then, $\begin{pmatrix} 4 \\ 1 \end{pmatrix} (a \quad b) = \begin{pmatrix} 20 & 12 \\ 5 & 3 \end{pmatrix}$

Or, $\begin{pmatrix} 4a & 4b \\ a & b \end{pmatrix} = \begin{pmatrix} 20 & 12 \\ 5 & 3 \end{pmatrix}$

By equality of matrices, a = 5 and b = 3. Hence the required matrix is$(5 \quad 3)$.

Example 6

If M = $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and I is an identity matrix of 2×2, prove that M^2- 2M-5I = O.

Solution: Here,

$M^2 = M \times M$

$= \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1+6 & 2+2 \\ 3+3 & 6+1 \end{pmatrix}$

$= \begin{pmatrix} 7 & 4 \\ 6 & 7 \end{pmatrix}$

$2M = 2\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 2 \end{pmatrix}$ and

$5I = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

Therefore,

$M^2 - 2M - 5I = \begin{pmatrix} 7 & 4 \\ 6 & 7 \end{pmatrix} - \left[\begin{pmatrix} 2 & 4 \\ 6 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right]$

$= \begin{pmatrix} 7 & 4 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} 7 & 4 \\ 6 & 7 \end{pmatrix}$

$= \begin{pmatrix} 7-7 & 4-4 \\ 6-6 & 7-7 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Hence, $M^2 - 2M - 5I = O$

c. Properties of Matrix Multiplication

In matrix algebra, multiplication of matrices do not follow exactly the rule that have been verified in algebraic multiplication of variable or constants. For example, in algebra $x \times y = xy$ and $y \times x = yx$ this implies $xy = yx$, but in matrices we have seen that $A \times B \neq B \times A$, and $A \times B = B \times A$ is true only when $A = B$ and both square matrices. Hence in this sub unit, we shall check on different properties of matrix.

i) Associative Property

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 8 \\ 7 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix}$

Now, $AB = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 5 & 8 \\ 7 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1\times 5+2\times 7 & 1\times 8+2\times 3 \\ 3\times 5+(-1)\times 7 & 3\times 8+(-1)\times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5+14 & 8+6 \\ 15-7 & 24-3 \end{pmatrix}$$

$$= \begin{pmatrix} 19 & 14 \\ 8 & 21 \end{pmatrix}$$

$$\therefore (AB)C = \begin{pmatrix} 19 & 14 \\ 8 & 21 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 19\times(-1)+14\times(-2) & 19\times(-2)+14\times 3 \\ 8\times(-1)+21\times(-2) & 8(-2)+21\times 3 \end{pmatrix}$$

$$\therefore \quad = \begin{pmatrix} -19-28 & -38+42 \\ -8-42 & -16+63 \end{pmatrix}$$

$$\therefore (AB)C = \begin{pmatrix} -47 & 4 \\ -50 & 47 \end{pmatrix} \text{..............(i)}$$

$$\text{Again, } BC = \begin{pmatrix} 5 & 8 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5\times(-1)+8\times(-2) & 5\times(-2)+8\times 3 \\ 7\times(-1)+3\times(-2) & 7\times(-2)+(3)\times(3) \end{pmatrix}$$

$$= \begin{pmatrix} -5-16 & -10+24 \\ -7-6 & -14+9 \end{pmatrix}$$

$$\therefore A(BC) = \begin{pmatrix} -21 & 14 \\ -13 & -5 \end{pmatrix}$$

$$\text{And } A(BC) = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -21 & 14 \\ -13 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times(-21)+2(-13) & 1\times 14+2(-5) \\ 3\times(-21)+(1)(-13) & 3\times 14+(-1)(-5) \end{pmatrix}$$

$$= \begin{pmatrix} -21-26 & 14-10 \\ -63+13 & 42+5 \end{pmatrix}$$

$$\therefore A(BC) = \begin{pmatrix} -47 & 4 \\ -50 & 47 \end{pmatrix} \text{........................(ii)}$$

From (i) and (ii)

(AB)C = A(BC).

Multiplication of matrices is associative; i.e. associative property holds true in matrix multiplication.

ii) Distributive Property

Let A = $\begin{pmatrix} 5 & 8 \\ 3 & -2 \end{pmatrix}$, B = $\begin{pmatrix} 2 & -7 \\ 4 & -6 \end{pmatrix}$ and C = $\begin{pmatrix} -5 & -3 \\ 1 & -2 \end{pmatrix}$.,

Now, B+C = $\begin{pmatrix} 2 & -7 \\ 4 & -6 \end{pmatrix} + \begin{pmatrix} -5 & -3 \\ 1 & -2 \end{pmatrix}$

or, B+C = $\begin{pmatrix} 2-5 & -7-3 \\ 4+1 & -6-2 \end{pmatrix}$

∴ B+C = $\begin{pmatrix} -3 & -10 \\ 5 & -8 \end{pmatrix}$

Then A(B+C) = $\begin{pmatrix} 5 & 8 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} -3 & -10 \\ 5 & -8 \end{pmatrix}$

or, A(B+C) = $\begin{pmatrix} 5\times(-3)+8\times5 & 5\times(-10)+8\times(-8) \\ 3\times(-3)+(-2)\times5 & 3\times(-10)+(-2)(-8) \end{pmatrix}$

or, A(B+C) = $\begin{pmatrix} -15+40 & -50-64 \\ -9-10 & -30+16 \end{pmatrix}$

∴ A(B+C) = $\begin{pmatrix} 25 & -114 \\ -19 & -14 \end{pmatrix}$……………(i)

Again,

AB = $\begin{pmatrix} 5 & 8 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 2 & -7 \\ 4 & -6 \end{pmatrix}$

Or, AB = $\begin{pmatrix} 5\times2+8\times4 & 5\times(-7)+8(-6) \\ 3\times2+(-2)\times4 & 3\times(-7)+(-2)\times(-6) \end{pmatrix}$

Or, AB = $\begin{pmatrix} 10+32 & -35-48 \\ 6-8 & -21+12 \end{pmatrix}$

∴AB = $\begin{pmatrix} 42 & -83 \\ -2 & -9 \end{pmatrix}$

And, AC = $\begin{pmatrix} 5 & 8 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} -5 & -3 \\ 1 & -2 \end{pmatrix}$

Or, AC = $\begin{pmatrix} 5(-5)+8\times1 & 5(-3)+8\times(-2) \\ 3(-5)+(-2)\times1 & 3(-3)+(-2)(-2) \end{pmatrix}$

Or, AC = $\begin{pmatrix} -25+8 & -15-16 \\ -15-2 & -9+4 \end{pmatrix}$

∴ AC = $\begin{pmatrix} -17 & -31 \\ -17 & -5 \end{pmatrix}$

Now, AB+AC = $\begin{pmatrix} 42 & -83 \\ -2 & -9 \end{pmatrix} + \begin{pmatrix} -17 & -31 \\ -17 & -5 \end{pmatrix}$

Or, AB+AC = $\begin{pmatrix} 42-17 & -83-31 \\ -2-17 & -9-5 \end{pmatrix}$

∴ AB+AC = $\begin{pmatrix} 25 & -114 \\ -19 & -14 \end{pmatrix}$……………………..(ii)

From (i) and (ii) we get,

A(B +C) = AB+AC.

Also, (A+B)×C = A×C + B×C (verify it)

Multiplication of matrix distributes over addition; i.e. distributive property holds true in matrix multiplication.

Similarly, A(B – C) = AB –AC is true.

iii) Identity Property

Let A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ be a square matrix and I be the identity matrix of order same as A, then,

AI $= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1\times1+2\times0 & 1\times0+2\times1 \\ 3\times1+4\times0 & 3\times0+4\times1 \end{pmatrix}$

$= \begin{pmatrix} 1+0 & 0+2 \\ 3+0 & 0+4 \end{pmatrix}$

∴AI = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ = A…………………(i)

And IA $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1\times1+0\times3 & 1\times2+0\times4 \\ 0\times1+1\times3 & 0\times2+1\times4 \end{pmatrix}$

$= \begin{pmatrix} 1+0 & 2+0 \\ 0+3 & 0+4 \end{pmatrix}$

∴IA = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ = A………………(ii)

Here, from (i) and (ii) it is seen that AI = A = IA.

The square matrix I acts as the identity matrix in matrix multiplication This means, identity matrix exists in matrix multiplication.

(iv) Property of Transpose of matrix product

Let, A = $\begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix}$

Now, AB $= \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix}$

$= \begin{pmatrix} 5 \times (-2) + 1 \times 4 & 5 \times 3 + 1 \times (-1) \\ 2 \times (-2) + 3 \times 4 & 2 \times 3 + 3 \times (-1) \end{pmatrix}$

$= \begin{pmatrix} -10 + 4 & 15 - 1 \\ -4 + 12 & 6 - 3 \end{pmatrix}$

∴ AB = $\begin{pmatrix} -6 & 14 \\ 8 & 3 \end{pmatrix}$

And $(AB)^T = \begin{pmatrix} -6 & 8 \\ 14 & 3 \end{pmatrix}$………………(i)

Again, $A^T = \begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix}$ and $B^T = \begin{pmatrix} -2 & 4 \\ 3 & -1 \end{pmatrix}$

Then, B^T x A^T $= \begin{pmatrix} -2 & 4 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix}$

$= \begin{pmatrix} (-2) \times 5 + 4 \times 1 & (-2) \times 2 + 4 \times 3 \\ 3 \times 5 + (-1) \times 1 & 3 \times 2 + (-1) \times 3 \end{pmatrix}$

$= \begin{pmatrix} -10 + 4 & -4 + 12 \\ 15 - 1 & 6 - 3 \end{pmatrix}$

∴ $B^TA^T = \begin{pmatrix} -6 & 8 \\ 14 & 3 \end{pmatrix}$

Hence $(AB)^T = B^TA^T$

This property is called the property of transpose of Matrix Product.

The transpose of product of two square matrices equals is to the product of their transpose matrices multiplied in opposite order.

Example 7

If I is the unit matrix of order 2×2, show that $I = I^2 = I^3 = I^4 = ………I^n$.

Solution:

Let I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then, $I^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1\times1+0\times0 & 1\times0+0\times1 \\ 0\times1+1\times0 & 0\times0+1\times1 \end{pmatrix}$

$\therefore I^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Now, $I^3 = I^2 \times I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as above.

This process could be continued till n^{th} power of I.

Hence, $I = I^2 = I^3 = I^4 = \dots\dots\dots I^n$.

If I is the unit matrix, then $I = I^2 = I^3 \dots I^n$. This property of identity matrix is called the idempotent property.

Exercise 3.5

1. If $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

   $C = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$, $D = \begin{pmatrix} p & q & r \\ w & x & y \end{pmatrix}$

   $E = \begin{pmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$, $F = \begin{pmatrix} \sqrt{2} & \sqrt{7} \\ \sqrt{3} & \sqrt{11} \\ \sqrt{5} & \sqrt{13} \end{pmatrix}$

   $G = \begin{pmatrix} 2 & 5 & 3 \end{pmatrix}$ and $H = \begin{pmatrix} 5 \\ 25 \end{pmatrix}$ are given matrices.

   a) Which of the pair of two matrices are confirmable for matrix multiplication. Justify with reason.

   b) Which of the matrices given above could by multiplied by itself like $A\times A = A^2$?

   c) Under which condition the cube of any matrix is defined as $A^3 = A\times A\times A$?

   d) Define scalar multiplication of matrix with suitable example.

   e) List out the properties of matrix multiplication.

2. If $A = \begin{pmatrix} 3 & -1 \\ 4 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$, find

   a) 7A　　　　f) A(B+C)

   b) 2A+3B　　　　g) $A^2-AB-BA+B^2$

c) 2A+3B-C h) A(B+C)

d) $(A+B)^2$ i) AB+AC

e) $A^2+AB+BA+B^2$ j) A(BC)

k) (AB)C l) If 3A+B+X = C, find the matrix X.

3. a) Carry out the multiplication of the following matrices.

i) $\begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ ii) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 5 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$

b) If A = $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$, B = $\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$ and C = $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, find

i) AB ii) BA iii) BC iv) CB

v) A^2 vi) B^2 vii) (A+B)C viiii) A(B+C)

c) If M = $\begin{pmatrix} 2 & 0 & 3 \\ -1 & 4 & 9 \end{pmatrix}$ and N = $\begin{pmatrix} -1 & 2 \\ 2 & 2 \\ -3 & -3 \end{pmatrix}$ work out MN and NM. What can you say about these two products?

d) If A = $\begin{pmatrix} 4 & -1 \\ -7 & -3 \end{pmatrix}$, B = $\begin{pmatrix} 2 & -5 \\ 2 & -5 \end{pmatrix}$ and C = $\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$, prove that

i) AB≠BA ii) A(B+C) = AB+AC

iii) A(B+C) ≠ AB+CA iv) $(AB)^T = B^TA^T$

v)A×I = I×A where I is 2×2 identity matrix.

4. a) If X = $\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$ and XY = $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ find the matrix Y.

b) If $\begin{pmatrix} -4 & 0 \\ 5 & 7 \end{pmatrix}(X) = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$ find the matrix X.

c) If $\begin{pmatrix} 2x+3 \\ y-2 \\ 3z+4 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 12 & 20 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix}$, find the value of x, y and z.

d) If P = $\begin{pmatrix} 4 & 9 \\ 16 & 25 \end{pmatrix}$ and PQ = $\begin{pmatrix} 4 & 9 \\ 16 & 25 \end{pmatrix}$, find the matrix Q.

e) If $\begin{pmatrix} 8 & -2 \\ 1 & 4 \end{pmatrix} A = 2\begin{pmatrix} 6 \\ 9 \end{pmatrix}$, find the matrix A.

f) If M = $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$, N = $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ and MN = M+N, find the values of x, y and z.

5. a) If X = $\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$, I and O are 2×2 identity matrices and zero matrix respectively, prove that X^2-6X+9I = 0.

b) If A = $\begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}$ and I is a 2×2 unit matrix, prove that (A+2I) (A-3I) = O where O is a 2×2 zero matrix.

c) If P = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ prove that.

i) P^2-2P = O   ii)$2P^2 = P^3$ where O is a zero matrix of order 2×2.

d) If A = $\begin{pmatrix} 3 & -5 \\ -4 & 2 \end{pmatrix}$, prove that A^2-5A = 14I. Where I is a unit matrix of order 2×2.

e) If A = $\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$, B = $\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$, C = $\begin{pmatrix} 6 & 8 \\ 7 & 9 \end{pmatrix}$ and I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, prove that.

i) AB≠BA   ii) A(BC) = (AB)C

iii) A(B+C) = AB+AC   (iv) IA = AI = A.

v. $(AB)^T = B^TA^T$ (vi) I^4 = I,

6. a) If $\begin{pmatrix} 10 & 8 \\ 2 & 4 \end{pmatrix}$X = $\begin{pmatrix} 18 \\ 6 \end{pmatrix}$, find X.

b) If X$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 7 \end{pmatrix}$, find X.

c) If $\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$, find x and y.

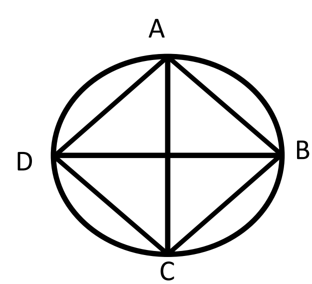
d) If x = $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, show that X^2 = 2X.

e) If A = $\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$, B = $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$and AB = A+B, find x, y and z.

f) If A = $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, prove that.

i) $(A+B)^T = A^T+B^T$ ii) $(AB)^T = B^TA^T$.

7. The adjoining figure shows that the places and routes of travel from one place to another. construct table to represent different routes of each place and write in terms of a matrix.



Unit 4

Coordinates Geometry

4.0 Review

Work in groups of students.

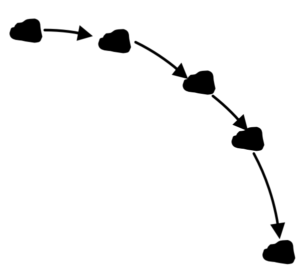
1. Divide all students in suitable groups. Ask all groups to plot any two points in XY-plane in a graph or grid paper. After that tell them to join those two points by using scale.

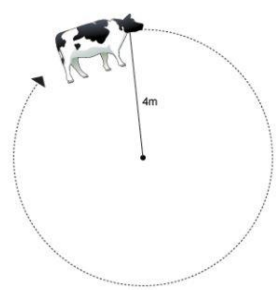
   Ask them to find the distance between these two points by using the formula.

   Distance = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)}^2$

2. Distribute different triangles and quadrilaterals (equilateral triangles, isosceles triangles, scalene triangles, square, rectangle, parallelogram and rhombus) with the coordinates of vertices. Then ask to verify the given geometric figure and their characteristics without measuring the length of sides or by using distance formula.

4.1 Locus





Observe the above figures and discuss about the path made by the moving object. The word locus is derived from Latin word. The set of all points that satisfy a given condition is called locus. It is the path made by a moving point under certain condition.

A locus is the set of points which satisfies a given condition. In other words, a locus is the path of a moving point under the given condition.

In above figure the first figure show the path obtained by a moving stone. In second figure a cow in a ground makes a circular path when it moves from fixed point. These all are the examples of locus.

How to represent a locus in co-ordinate geometry?

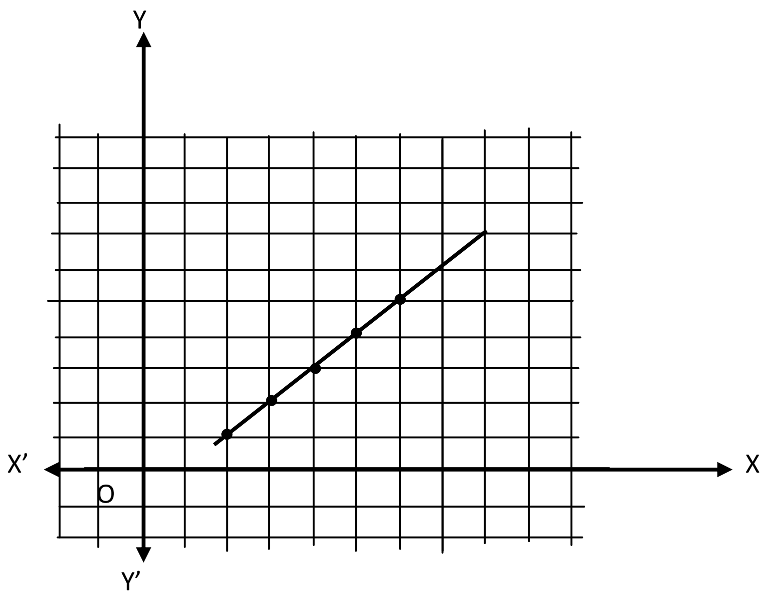
The set of points under the given conditions can be represented in terms of Cartesian coordinates and the *locus* is analytically defined by an equations.

Equation of the locus:

The values of *x* and *y* for a locus are given as follows.

x	2	3	4	5	6
y	1	2	3	4	5

Plotting these points on Cartesian coordinate, we get a straight line.



Here in each case y coordinates is 1 less than x-coordinate, so the relation of x and y can be expressed as $y = x-1$.

i.e. The locus of a point is $x-y-1 = 0$

How to find locus of a point?

To find the equation of locus, the following procedure should be followed.

- Assume that (x, y) lies on the locus.
- Write algebraic condition that (x, y) satisfy.
- Express the conditions in terms of x and y.
- Simplify the algebraic expression.

Note: If any point lies on the locus its coordinates must satisfy the equation of locus.

Example 1

Does the point (0, 5) lies on $x^2+y^2 = 25$?

Solution:

Here, $(x, y) = (0, 5)$

Equation of locus is $x^2+y^2 = 25$ ………… (i)

By putting (0, 5) in (i) we get,

$0^2 + 5^2 = 25$

or, $25 = 25$ (satisfied)

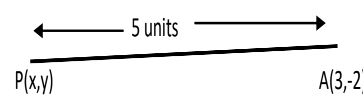
Hence, (0, 5) lies on the locus $x^2+y^2 = 25$

Example 2

Find the equation of a locus of a point which moves such that its distance from (3, -2) is always 5 units.

Solution:

Here, A (3, -2) be a given point and let P (x, y) be a point on the locus such that d(PA) = 5 units



We know that

Distance d(PA) $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

or, 5 units $= \sqrt{(3 - x)^2 + (-2 - y)^2}$

On squaring both side we get,

or $(3-x)^2+(-2-y)^2 = 25$

or $9 - 6x+x^2+4+4y+y^2 = 25$

or, $x^2+y^2-6x+4y+13 = 25$

$\therefore x^2+y^2-6x+4y-12 = 0$ is required equation of locus.

Example 3

Find the equation of locus of a point P which moves such that its distance from L(-4, 3) and M (1, 0) satisfies the condition $PL^2 = PM^2$.

Solution:

We have given, L(-4, 3) and M (1, 0). Let P (x, y) be any point on the locus By given condition, $(PL)^2 = (PM)^2$

$$or, \quad (x+4)^2 + (y-3)^2 = (x-1)^2 + (y-0)^2$$
$$or, \quad x^2 + 8x + 16 + y^2 - 6y + 9 = x^2 - 2x + 1 + y^2$$
$$or, \quad 8x - 6y + 2x + 25 - 1 = 0$$
$$or, \quad 10x - 6y + 24 = 0$$
$$or, \quad 5x - 3y + 12 = 0$$

Therefore, 5x – 3y +12 = 0 is required equation of locus.

Exercise 4.1

1. **Find the locus of P (x, y) which moves such that**
   a) Its distance from (-4, 5) is 5.
   b) Its distance form *x-axis is always 5 units.*
   c) Its distance from y-axis is always – 3.
   d) Its distance from (-5, -2) is always 6 units.
   e) Its distance from origin is 3.
   f) Its distance from (1, 6) is 7

2. a) **Find which of the following points lie in $x^2+y^2+2x+4y-8 = 0$**
   i) (1, 1) ii) (–1, 2) iii) (3, 1)

   b) **Which of the points below lie on $x^2+y^2+10x+4y-32 = 0$**
   i) (1, 3) ii) (2, –3)

   c) **Does the point (3, 4) lie on the loci given below.**
   i) $x^2+y^2 = 25$ ii) $2x+3y = 12$
   iii) $3x+4y = 25$ iv) $2x+2y+3 = 0$

   d) If (4, 4) lies on the locus $y^2 = ax$, prove that (16, 8) lies on the same locus.

   e) If (2, -3) lies on the locus $kx^2+3y^2+2x-6 = 0$, find the value of k.

   f) If (0, 4) and (4, 0) both lie on $\frac{x}{a} + \frac{y}{b} = 1$ find the values of a and b.

3. a) Find the equation of locus of a point which moves in such a way that its distances from (a, b) is k.

   b) Find the equation of the locus of a point lying equidistant from

i) (0, 2) and *x*-axis ii) (3, 5) and (6, 0)

iii) (2, –3) and (–1, 8) iv) (–2, 7) and (5, 6)

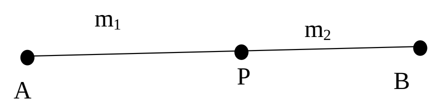
v) both the axes

c) Let A (5, 0) and B (5, 0) be two fixed points. Find the locus satisfying the following conditions:

a) $PA^2+PB^2 = AB^2$ b) PA = 2PB c) PA:PB = 2:3

4.2 Section formula

Let AB be a line segment and P be any point lying on the line segment such that the point P divides AB into two segments AP and PB.



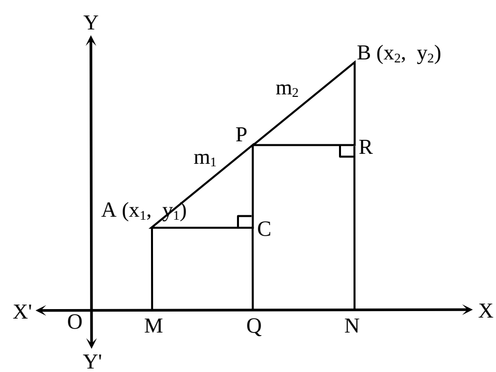
In this case $\frac{AP}{PB} = \frac{m_1}{m_2}$

What will be the coordinates of P? Discuss.

Internal division of a line segment

To find the coordinates of a point that divides the given line segment in the given ratio m_1:m_2

Let P(x, y) be a point on the line joining A(x_1, y_1) and B(x_2, y_2). Let the point P divides AB in the ratio of m_1:m_2.



Draw AM, PQ and BN perpendiculars on x-axis,

Also draw AC⊥PQ and PR⊥BN.

From the figure,

$OM = x_1$ $AM = QC = y_1$

$OQ = x$ $PQ = NR = y$

$ON = x_2$ $BN = y_2$

$PC = PQ – QC = y – y_1$

$AC = MQ = OQ – MO = x – x_1$

$BR = BN – NR = y_2 – y$

$PR = QN = ON – OQ = x_2 – x$

In right angled triangles APC and BPR,

(i) $\angle ACP = \angle PRB = 90^o$

(ii) $\angle PAC = \angle BPR$ (Corresponding angles)

$\Delta APC \sim \Delta PRB$ (By AA similarity)

$\frac{AP}{PB} = \frac{AC}{PR} = \frac{PC}{BR}$ (Ratio of corresponding sides of similar triangles)

or, $\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$ ………………..(i)

From first and second ratios of (i)

$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x}$

or, $m_1x_2 - m_1x = m_2x - m_2x_1$.

$m_1x + m_2x = m_1x_2 + m_2x_1$

or, $(m_1 + m_2)x = m_1x_2 + m_2x_1$

or, $x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$

Again taking first and third ratio of (i)

$\frac{m_1}{m_2} = \frac{y - y_1}{y_2 - y}$

or, $m_1y_2 - m_1y = m_2y - m_2y_1$

or, $m_1y_2 + m_2y_1 = m_1y + m_2y$

or, $m_1y_2 + m_2y_1 = (m_1 + m_2)y$

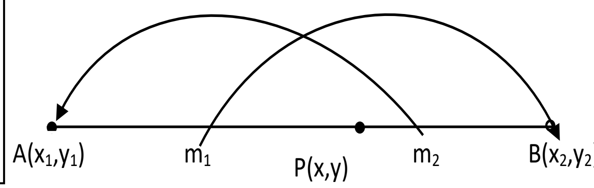
or, $y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$

$\therefore$ The coordinates of $P(x, y) = P\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$

$\therefore$ The coordinates of point P that divides the line segments in the ratio $m_1:m_2$ is $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$

For memory technique we can use the following idea:

m_1 is multiplies by x_2 (or y_2) and m_2 is multiplied by x_1 (or y_1) and their sum is divided by sum of m_1 and m_2.



Example 1

Find the coordinates of the point P (x, y) that divides the line segment joining A (2, 3) and B (7, 8) internally in the ratio 2:3.

Solution:

We have, $x_1 = 2$, $x_2 = 7$, $y_1 = 3$, $y_2 = 8$

Now, $m_1 = 2$ and $m_2 = 3$

Since P (x, y) divides AB in the ratio 2:3 then

$$x = \frac{m_1x_2+m_2x_1}{m_1+m_2} \text{ and } y = \frac{m_1y_2+m_2y_1}{m_1+m_2}$$

$$\text{or } x = \frac{2\times7+3\times2}{2+3} \text{ and } y = \frac{2\times8+3\times3}{2+3}$$

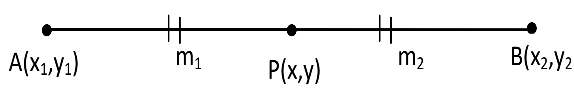
$$x = \frac{20}{5} = 4 \quad \text{and} \quad y = \frac{25}{5} = 5$$

The required point is P (4, 5).

Midpoint formula of a line segment

If the point P (x, y) divides the line segment AB in two equal parts then $m_1 = m_2$ and the coordinates of point P is

$$= \left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \quad \frac{m_1y_2+m_2y_1}{m_1+m_2}\right)$$



$$= \left(\frac{m_1x_2+m_1x_1}{m_1+m_1}, \quad \frac{m_1y_2+m_1y_1}{m_1+m_1}\right)$$

$$= \left(\frac{m_1(x_2+x_1)}{2m_1}, \quad \frac{m_1(y_2+y_1)}{2m_1}\right)$$

$$= \left(\frac{x_1+x_2}{2}, \quad \frac{y_1+y_2}{2}\right)$$

If P (x, y) be the midpoint of line segment joining A (x_1, y_1) and B(x_2, y_2), then the coordinates of P is $\left(\frac{x_1+x_2}{2}, \quad \frac{y_1+y_2}{2}\right)$

Example 2

Find coordinates of a point which divides the line segment joining C(4, –12) and D(6, 8) into two equal parts.

Solution:

Let P(x, y) be the point that divides the line segment joining C (4, –12) and D(6, 8) into two equal parts. Then P is midpoint of line segment CD and the coordinates of mid-point is

$x = \frac{x_1+x_2}{2}$, $y = \frac{y_1+y_2}{2}$

or, $x = \frac{6+4}{2}$, $y = \frac{-12+8}{2}$

or, x = 5   y = - 2

The coordinates of midpoint is (5, –2)

Example 3

Find midpoint of line segment joining (4, 6) and (–6, –4).

Solution: Here,

$(x_1, y_1) = (4, 6)$ and $(x_2, y_2) = (-6, -4)$

(x, y) = ?

We have $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$

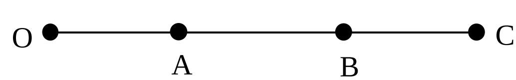
or, $x = \frac{4 + (-6)}{2}$, $y = \frac{6 + (-4)}{2}$

or, $x = \frac{-2}{2}$, $y = \frac{2}{2}$

∴ x = –1, y = 1

∴ (–1, 1) is midpoint of the given line segment.

External Division of a Line Segment



Let Enjal and Enjila start to walk from a fixed point O for the point C. After some time

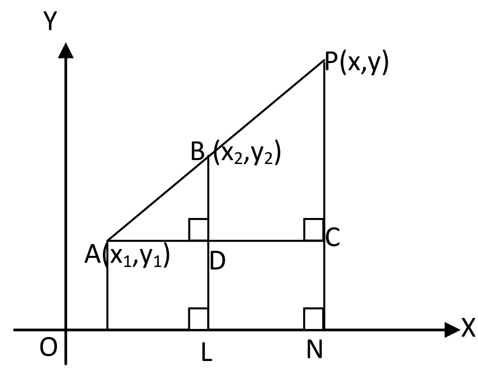
Enjal and Enjila reached at the points A and B respectively. Enjal have to walk 8 km and Enjila has to walk 6 km to cover the all distance then the ratio of AC and BC is 8: 6

i.e.$\frac{AC}{BC} = \frac{8}{6} = \frac{4}{3}$

Enjal has to cover $\frac{4}{3}$ of Enjila

In this case point C is said to divide the segment AB externally in the ratio of 4:3.

To find the coordinates of a point that divides the line segment joining (x_1, y_1) and (x_2, y_2) externally.



Let A (x_1, y_1) and B (x_2, y_2) be two points. Let P (x, y) divides the line segment AB externally in the ratio $m_1:m_2$.

ie, AP : PB = $m_1 : m_2$

Draw AL, BM and PN perpendiculars on OX (x-axis) and

AC⊥PN which meets BM at point D.

AL || BM since BM || PN.

From figure, AD = LM = OM – OL = $x_2 - x_1$, AB = AP – BP

AC = LN = ON – OL = $x - x_1$

BD = BM – MD = $y_2 - y_1$

PC = PN – CN = $y - y_1$

Since ΔABD ~ ΔAPC, we have

$$\frac{AB}{AP} = \frac{AD}{AC} = \frac{BD}{PC} \text{ ………………… (i)}$$

Now, $\frac{AB}{PB} = \frac{m_1}{m_2}$

or, $\frac{PB}{AB} = \frac{m_2}{m_1}$

or, $\frac{AP-AB}{AP} = \frac{m_2}{m_1}$

or, $\frac{AP}{AP} - \frac{AB}{AP} = \frac{m_2}{m_1}$

or, $1 - \frac{AB}{AP} = \frac{m_2}{m_1}$

or, $\frac{\mathrm{AB}}{\mathrm{AP}} = 1 - \frac{\mathrm{m_2}}{\mathrm{m_1}} = \frac{\mathrm{m_1 - m_2}}{\mathrm{m_1}}$

Substituting the values above, we get

or, $\frac{m_1 - m_2}{m_1} = \frac{x_2 - x_1}{x - x_1} = \frac{y_2 - y_1}{y - y_1}$…………………(ii)

By taking first and second ratio from (ii)

$$\frac{m_1 - m_2}{m_1} = \frac{x_2 - x_1}{x - x_1}$$

or, $(m_1 - m_2)(x - x_1) = m_1 x_2 - m_1 x_1$

or, $(m_1 - m_2)x - m_1 x_1 + m_2 x_1 = m_1 x_2 - m_1 x_1$

or, $(m_1 - m_2)x = m_1 x_2 - m_2 x_1$

or, $x = \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}$

Similarly taking first and third ratio from equation (i)

$$\frac{m_1 - m_2}{m_1} = \frac{y_2 - y_1}{y - y_1}$$

$(m_1 - m_2)(y - y_1) = m_1 y_2 - m_1 y_1$

or, $(m_1 - m_2)\, y - m_1 y_1 + m_2 y_1 = m_1 y_2 - m_1 y_1$

or, $y = \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}$

The coordinates of external divisor P is $\left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \quad \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}\right)$

Example 4

Find the coordinates of a point that divides the line segment joining (6, –2) and (–3, 4) externally in the ratio of 5:2.

Solution:

We have $(x_1, y_1) = (6, 2)$

$(x_2, y_2) = (-3, 4)$

$m_1 = 5, m_2 = 2$

Let P (*x*, *y*) be coordinates of the point that divides the segment in given ratio, then

$x = \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}$ and $y = \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}$

or, $x = \frac{5\times-3-2\times6}{5-2}$, and $y = \frac{5\times4-2\times-2}{5-2}$

or, $x = \frac{-27}{3}$, *and* $y = \frac{24}{3}$

or, $x = -9$, $y = 8$

$P(x, y) = (-9, 8)$

Example 5

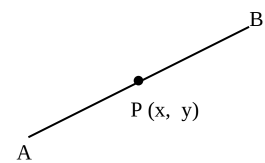
Find the coordinates of a point which divides the line joining (6, –3) and (–1, 4) in the ratio of 3 : 4 (i) internally (ii) externally.

Solution:

Here, $A(x_1, y_1) = (6, -3)$ $B(x_2, y_2) = (-1, 4)$

$m_1 : m_2 = 3 : 4$

$(x, y) = ?$



Now, (i) for internal divisor

$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$ and $y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$

$= \frac{3\times-1+4\times6}{3-4}$ and $y = \frac{3\times4+4\times-3}{3+4}$

$= \frac{-3+24}{7}$ $= \frac{12-12}{7}$

$= \frac{21}{7} = 3$ $= 0$

$\therefore$ (3, 0) divides AB in the ratio of 3:4 internally.

(ii) For external divisor

$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}$, $y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$

$x = \frac{3\times(-1)-4\times6}{3-4}$, $y = \frac{3\times4-4\times-3}{m_1-m_2}$

$x = \frac{-3-24}{-1}$, $y = \frac{12+12}{3-4}$

$= \frac{-27}{-1}$, $y = \frac{24}{-1}$

$= \quad 27 \qquad\qquad = -24$

$\therefore$ P(27, –24) divides externally in the ratio 3:4.

Example 6

Find the coordinates of the point dividing the line joining point (5, –2) and (9, 6) in the ration of 3:1 (i) internally (ii) externally

Solution: We have,

$$x_1 = 5, \quad y_1 = -2, \quad x_2 = 9, \quad y_2 = 6$$

$$m_1 = 3, \quad m_2 = 1$$

(i) Let P(x, y) divides the line internally. So,

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \text{ and } \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

$$x = \frac{3 \times 9 + 1 \times 5}{3 + 1}, \quad and \quad y = \frac{3 \times 6 + 1 \times -2}{3 + 1}$$

$$x = \frac{32}{4}, \quad and \quad y = \frac{16}{4}$$

$$x = 8, \quad and \quad y = 4$$

$\therefore$ (x, y) = (8, 4)

(ii) Let P'(x, y) divides the line externally then

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \text{ and } \quad y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

or, $x = \frac{3\times9-1\times5}{3-1}$, and $\quad y = \frac{3\times6-1\times-2}{3-1}$

or, $x = \frac{22}{2}$, and $\quad y = \frac{20}{2}$

or, $x = 11$, and $\quad y = 10$

$\therefore$ P' (x, y) = (11, 10)

Example 7

Find the ratio in which point P (1, 4) divides the line segment joining the points A (–1, 6) and B (2, 3) internally.

Solution: Method I

Suppose the ratio is $m_1 : m_2$.

Then by using formula for $x = 1, y = 4, x_1 = -1, y_1 = 6, x_2 = 2, y_2 = 3$ we get

$$x = \frac{m_1x_2+m_2x_1}{m_1+m_2}$$

or, $1 = \frac{m_1\times 2+m_2(-1)}{m_1+m_2}$

or, $m_1 + m_2 = 2m_1 - m_2$

or, $2m_{1\_}m_1 = 2m_2$

$or,\ m_1 = 2m_2$

or $\frac{m_1}{m_2} = \frac{2}{1}$

$\therefore$ The required ratio is 2:1

Similarly, by using the formula for y as;

$y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$ we can get same result.

Method II

Also, if we take the ratio is k:1 instead of m_1:m_2, we get the same ratio as follows:

Let, $x = \frac{m_1x_2+m_2m_2}{m_1+m_2}$

or, $1 = \frac{k \times 2 + 1 \times (-1)}{k + 1}$

or, $k + 1 = 2k - 1$

or, $2k - k = 1 + 1$

$k = 2$

$\therefore$ The ratio is k:1 = 2:1

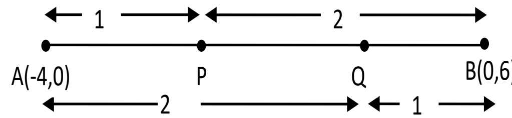
Example 8

Find the coordinates of points which divides the line segment joining the points (-4, 0) and (0, 6) in three equal parts.

Solution:

Suppose P and Q be the two points which divides the line segment joining (-4, 0) and (0, 6) in three equal parts. Then P divides the line in 1:2 ratio and Q divides the line in 2:1 ratio

For P, P divides in 1:2 ratio



$$x = \frac{m_1x_2+m_2x_1}{m_1+m_2}, \qquad y = \frac{m_1y_2+m_2y_1}{m_1+m_2}$$

$$= \frac{1.0+2.(-4)}{1+2}, \qquad = \frac{1.6+2.0}{1+2}$$

$$= \frac{-8}{3} \qquad = \frac{6}{3} = 2$$

$$= \left(\frac{-8}{3}, 2\right)$$

For Q, Q divides AB in 2:1 ratio.

$$(x, y) = \frac{m_1x_2+m_2x_1}{m_1+m_2}, \quad \frac{m_1y_2+m_2y_1}{m_1+m_2}$$

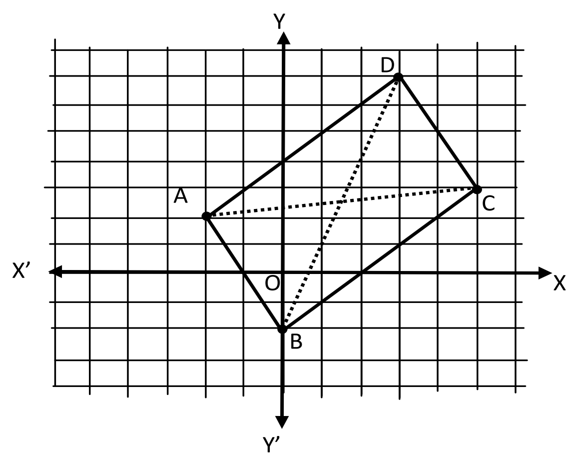
$$= \left(\frac{2\times0+1\times-4}{2+1}, \quad \frac{2\times6+1.0}{2+1}\right)$$

$$= \left(\frac{-4}{3}, \frac{12}{3}\right) = \left(\frac{-4}{3}, 4\right)$$

Example 9

Prove that A (–2, 2), B (0, –2), C(5, 3) and D (5, 7) are the vertices of a parallelogram ABCD.

Solution:



We have the given four points as A(–2, 2), B (0, –2), C(5, 3), D(3, 7). where, AC and BD are diagonals. Now the midpoint of A (–2, 2) and C (5, 3) is

$$\left(\frac{-2+5}{2}, \frac{2+3}{2}\right) = \left(\frac{3}{2}, \frac{5}{2}\right)$$

Again the midpoint of (0, –2) and D (3, 7) is

$$\left(\frac{0+3}{2}, \frac{-2+7}{2}\right) = \left(\frac{3}{2}, \frac{5}{2}\right)$$

Since the mid points of diagonals AC and BD are same $\left(\frac{3}{2}, \frac{5}{2}\right)$.

So ABCD is a parallelogram.

Example 10

P(3, 4), Q(–2, 1) and R(–5, 6) be the coordinates of the mid points of the sides AB, BC and CA of a ΔABC respectively. Find the coordinates of the vertices A, B, C of the triangle.

Solution

P (3, 4), Q (–2, 1) and R(–5, 6) are the mid points of sides AB, BC and AC with coordinates A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3).

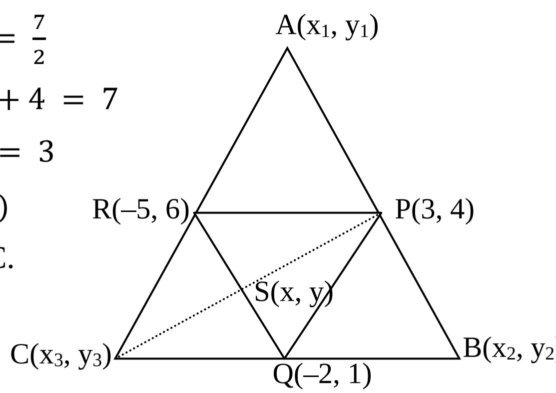
Now, from figure

PQCR is a parallelogram and S(x, y) is common midpoint of diagonals PC and QR.

Therefore, $\left(\frac{x_3+3}{2},\ \frac{y_3+4}{2}\right) = \left(\frac{-5-2}{2},\ \frac{6+1}{2}\right)$

or, $\left(\frac{x_3+3}{2},\ \frac{y_3+4}{2}\right) = \left(\frac{-7}{2},\ \frac{7}{2}\right)$

Now, equating the coordinates we get,



or, $\frac{x_3+3}{2} = \frac{-7}{2}$ and $\frac{y_3+4}{2} = \frac{7}{2}$

or, $x_3 + 3 = -7$ and $y_3 + 4 = 7$

or, $x_3 = -10$ and $y_3 = 3$

$\therefore$ C(x_3, y_3) = C(-10, 3)

Since R(-5, 6) is midpoints of AC.

So, $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$ = (–5, 6)

or, $\left(\frac{x_1-10}{2}, \frac{y_1+3}{2}\right)$ = (–5, 6)

$\therefore$ $\frac{x_1-10}{2}$ = –5 and $\frac{y_1+3}{2}$ = 6

or, x_1 = –10 + 10 and y_1 = 12 – 3

or x_1 = 0, y_1 = 0

$\therefore$ (x_1, y_1) = (0, 9)

Again, Q (–2, 1) is the midpoint of BC.

So, $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$ = (–2, 1)

or, $\left(\frac{x_2-10}{2}, \frac{y_2+3}{2}\right)$ = (–2, 1)

$\therefore \frac{x_2 - 10}{2} = -2$ and $\frac{y_2 + 3}{2} = 1$

or, $x_2 = -4 + 10$ and $y_2 = 2 - 3$

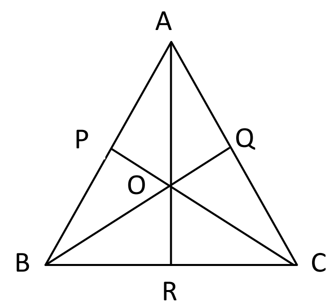
or, $x_2 = 6$ and $y_2 = 1$

$\therefore (x_2, y_2) = (6, -1)$

∴ The coordinates of the vertices of ΔABC are A(0, 9), B(6, –1) and C(–10, 3) /

Centroid of a triangle

The line joining the midpoint of a side and the opposite vertex of the triangle is called median of the triangle. There are three medians of a triangle.



The point of intersection of these three medians is called Centroid of the triangle.

The centroid divides each median in the ratio of 2:1 from the vertex to the midpoint.

In the given figure, O is centroid of ΔABC

Therefore, BO:OQ = CO:OP = AO:OR = 2:1

The coordinates of O (x, y) can be determined by

$$x = \frac{x_1 + x_2 + x_3}{3}, \qquad y = \frac{y_1 + y_2 + y_3}{3}$$

Example 11

If A (1, 1), B(1, 7) and C (7, 1) be the coordinates of vertices of ΔABC then find the centroid of ΔABC.

Solution: Here

A (1, 1), B(1, 7) and C (7, 1) be the coordinates of vertices of ΔABC, then the centroid of ΔABC has coordinates

$$x = \frac{x_1+x_2+x_3}{3}, \qquad y = \frac{y_1+y_2+y_3}{3}$$

$$\text{or, } x = \frac{1+1+7}{3}, \qquad y = \frac{1+7+1}{3}$$

$$\text{or, } (x, y) = \left(\frac{9}{3}, \frac{9}{3}\right) = (3, 3)$$

Exercise 4.2

1. **Find distance between following pair of points.**

   (a) (o, 3) and (4, 0) (b) (–5, 2) and (3, 8)

   (c) (7, 2) and (5, 4) (d) (3, 2) and (–3, –5)

2. **Find the coordinates of P in the followings cases.**

   (a) P (*x*, *y*) which divides the line segment joining (–1, 2) and (4, –5) internally in the ratio 2 : 3.

   (b) P (*x*, *y*) which divides the line segment joining (3, 9) and (1, –3) internally in the ratio 2 : 3.

   (c) P (*x*, *y*) which divides the line segment joining (–1, 3) and (8, 7) internally in the ratio 2 : 5.

   (d) P (*x*, *y*) which divides the line segment joining (2, –4) and (–5, 8) internally in the ratio 2 : 3.

3. **Find the coordinates of a point which divides the line segment joining the following points externally in the given ratio.**

   (a) A (–1, 1) and B (8, 9) in the ratio 3 : 4.

   (b) A (0, –5) and B (5, 10) in the ratio 2 : 1.

   (c) A (–3, 9) and B (1, –3) in the ratio 2 : 3.

   (d) P (–1, –3) and Q (8, 7) in the ratio 2 : 5.

4. **Find mid points of the line segment joining the following points.**

   (a) (2, 5) and (4, 4) (b) (0, 7) and (6, 3)

   (c) (–10, 6) and (2, –4) (d) (–2, –1) and (4, 3)

   (e) (3, –5) and (9, –3)

5. (a) Find the ratio of internal division of the line segment joining the points (5, –3) and (–9, 4) by the point (3, –4).

   (b) In what ratio does the point (15, 11) divides the line segment joining the points (5, 15) and (20, 9)?

   (c) Find the ratio in which the line segment joining (6, 21) and (1, –7) is divided internally by (*x*, *0*). Also find the value of x.

   (d) Find the ratio that the line segment joining the points (2, –4) and (5, 8) divided by the *x*-axis.

(e) Find the ratio that the line segment joining the points (–2, 4) and (5, 4) divided by y-axis. Also find value of *y*.

6. (a) Find the coordinates of the points which divides the line segment joining the points (–8, 0) and (4, –8) in four equal parts.

(b) Find the coordinates of the points of trisection of the line segment joining the points A (–3, 9) and B (6, 3).

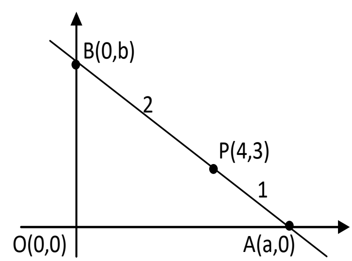
(c) Show that P (–2, 1) is a point of trisection of line segment joining the points A (1, 2) and B (–8, 5).

(d) Show that P (–2, 3) is point of trisection of line segment joining the points A (4, –5) and B (–6, 15).

7. (a) If P (–3, 3) divides the line segment joining A(x, 0) and B (0, y) in the ratio 2 : 3, then find coordinates of A and B.

(b) What will be the coordinates of points A on x-axis and B on y-axis if the point P (4, 5) intersects the line segment AB in the ratio 5 : 3?

(c) Find the value of a and b by using the information in the adjoining figure.

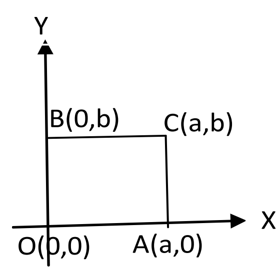


8. (a) Show that A (-4, 9), B (6, 9), C (7, 0) and D (–3, 0) are the vertices of a parallelogram.

(b) Show that the midpoint of line joining A (5, 7), and B (3, 9) is equal to the midpoint of line segment joining the points C (8, 6) and D (0, 10). Also write the name of quadrilateral thus formed.

(c) If ABCD be a parallelogram with vertices A(10, 6), B (0, –1), C (2, –5) and D (x, y). Find coordinates of D.

(d) In adjoining figure the coordinates of O, A, C and B are given. Show that OACB is a rectangle.



(e) Prove that A (1, –1), B (–2, 2), C (4, 8) and D (7, 5) are the vertices of a rectangle.

(f) If P (2, 1), Q (–2, 3) and R (4, 5) be the three vertices of a parallelogram PQRS, find coordinates of S opposite to Q.

(g) If A (1, 1), B (7, –3), C (12, 2) and D (7, 21) are the four vertices of a quadrilateral, prove that the mid points of segments AB, BC, CD and AD forms a parallelogram.

9. (a) Find centroid of the triangles whose vertices are

i) (4, 8), (–9, 7) and (8, 13)

ii) (3, –5), (7, 4) and (10, –2)

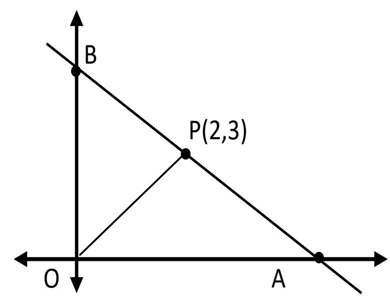
iii) (2, 1), (5, 2) and (3, 7)

(b) If two vertices of a triangle are (–3, 1) and (0, –2) and the centroid is (0, 0), then find third vertex of the triangle.

(c) If P (x, 7), Q (5, –2) and R (0, y) be three vertices and O (0, 0) be the centroid of a triangle PQR, find value of x and y.

(d) If P (4, –2), Q (–2, 3) and R (6, 4) are the vertices of a triangle, find the length of median drawn from Q to PR.

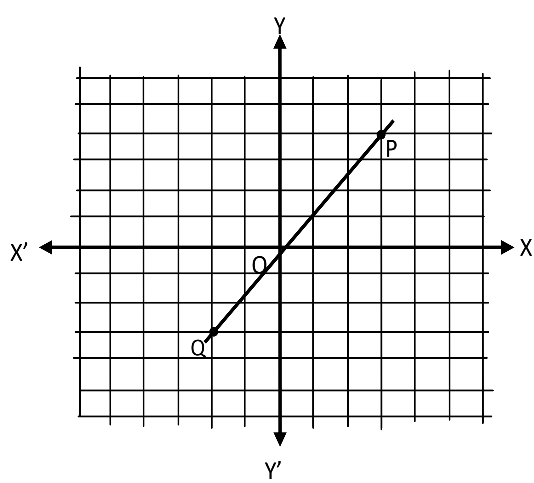
(e) If P (2, 3) is midpoint of the line segment, find coordinates of A and B and show that OP = AP = BP.



4.3 Equation of Straight Lines

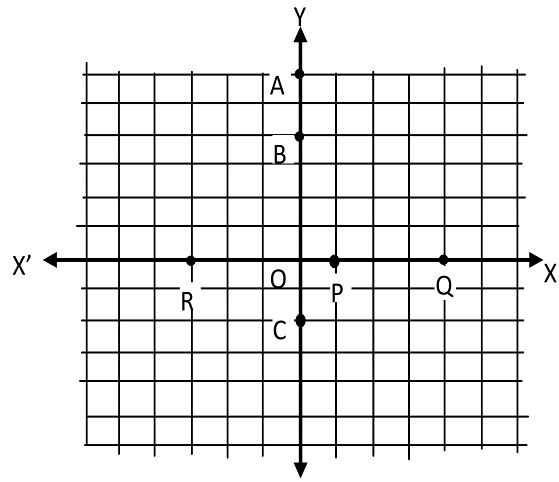
Take any two points on a cartesian plane. Join these points by using scale, what will we get discuss in group about the figure.

In figure points P(3, 4) and Q(-2, -3) are joined and line segment PQ is formed. This is a type of straight line. How many different kinds of straight lines can be formed from two points? Discuss in group of two.



Equations of the Coordinates Axes

(a) Equation of X-axis



What will be the coordinates of the point P, Q and R in given figure? They are P(1, 0), Q(4, 0), R(–3, 0) . In each cases the value of y is always 0.

Also each above point lie in x-axis. Hence, all over the X -axis the value of y is always zero.

The equation of X-axis is y = 0.

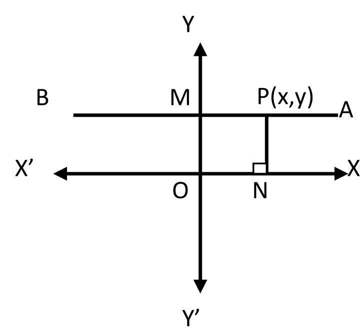
(b) Equation of Y-axis

The coordinates of A, B and C are (0, 6), (0, 4) and (0, -2) respectively. In each case the value of x is zero and the above all points lie in y-axis. So, All over the y-axis the value of x is always zero.

The equation of Y-axis is $x = 0$.

(c) Equation of a straight line parallel to X-axis

Let AB be a line parallel to X'OX. Which intersects y-axis at M such that OM = b.



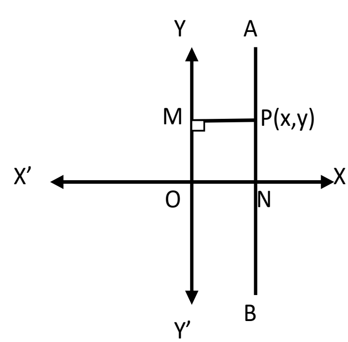
Let P (x, y) be any point on AB. Draw PN ⊥ OX then OMPN is a rectangle. So PN = OM

Since PN = y and OM = b

Hence, at point P(x, y) the value of y is b. Since P (x, y) be arbitrary point on AB so, throughout the line AB, we can say the value of y is equal to b i.e. y = b.

The equation of a straight line parallel to X-axis is y – b = 0.

(d) Equation of a straight line parallel to Y-axis



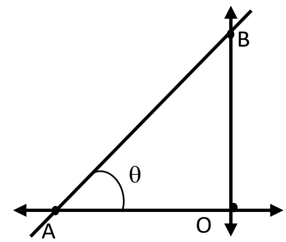
Let AB be a line parallel to YOY' (Y-axis) which intersects x–axis at N such that ON = a. Let P(x, y) be any point on AB. Draw PM⊥OY, then ON = PM = x, also ON = a

Hence, at point P(x, y) the value of x is a.

x–a = 0 is the equation of line AB. Since P(x, y) is arbitrary point on AB and AB is parallel to y-axis, the value of x throughout line AB is equal to a.

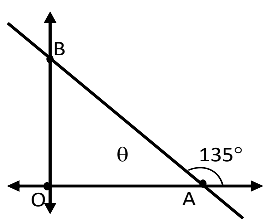
The equation of a straight line parallel to Y-axis is x – a = 0.

(e) Slope or Gradient of a straight line

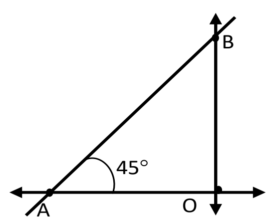


In figure, a straight line intersects the x-axis and y-axis at point A and B respectively then ΔAOB is a right angled triangle. The angle BAO, is the angle made by AB in positive x-axis (anticlockwise direction) is called inclination of AB.

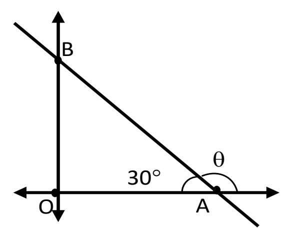
The slope of a straight line (gradient of straight line) is the tangent of inclination of the line. The slope of a line is denoted by m. If θ be the angle made by a line with x-axis in positive direction, then the slope is given by m = tanθ. For example, the slope of the following straight lines are:



(i)



(ii)



(iii)

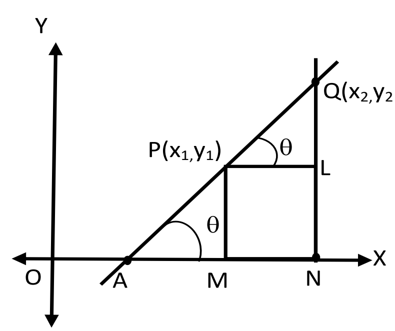
(i) m = tanθ = tan135° = –1

(ii) m = tanθ = tan45° = 1

(iii) $m = \tan(180° - 30°) = \tan 150° = -\frac{1}{\sqrt{3}}$

(f) Slope of a Straight Line Joining two Points

Let P (x_1, y_1) and Q (x_2, y_2) be two points and θ be the angle made by PQ with positive X-axis. Draw PM and QN perpendiculars on X-axis and PL ⊥ QN.



In figure,

PL = MN = ON – OM = $x_2 - x_1$

QL = QN – LN = $y_2 - y_1$

Since PL||OX so, ∠QPL = ∠QAN = θ

In right angled triangle QPL,

$\tan\theta = \frac{QL}{PL} = \frac{y_2 - y_1}{x_2 - x_1}$

Hence the slope of a straight line joining (x_1, y_1) and (x_2, y_2) is m $= \frac{y_2 - y_1}{x_2 - x_1}$.

Instruction:

To find slope of the straight line we first recall the values of fundamental angles of tan θ such as

θ→	0°	30°	45°	60°	90°
tanθ→	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Example 1

Find the slope of a straight line whose angle of inclination is 30°.

Solution: Here, we have

inclination (θ) = 30°

Slope (m) = ?

Now, slope (m) = tan θ

= tan 30°

$\therefore$ m = $\frac{1}{\sqrt{3}}$

Example 2

What will be the inclination of a line with slope 1?

Solution: Here, we have

slope (m) = 1

Angle of inclination (θ) = ?

Now, slope (m) = 1

or, tan θ = 1

tan θ = tan 45°

∴ θ = 45°.

∴ The inclination of the line is 45°.

Example 3

What will be the slope of a line joining the points P (4, 7) and Q (3, 4)?

Solution: Here, we have

P (x_1, y_1) = (4, 7), Q (x_2, y_2) = (3, 4)

Slope (m) = ?

We know that, slope (m) $= \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{4-7}{3-4} = \frac{-3}{-1} = 3$$

∴ Slope (m) = 3.

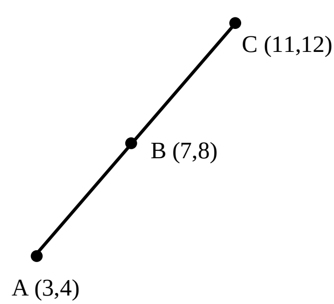
Collinear Points: Three points A, B, and C are said to be collinear if slope of AB and BC or slope of AB and AC are equal.

Example 4

Show that A (3, 4), B(7, 8) and C (11, 12) are collinear points.

Solution: Here,

A (3, 4), B(7, 8) and C (11, 12) are given .



We have,

Slope of AB $= \frac{y_2 - y_1}{x_2 - x_1}$

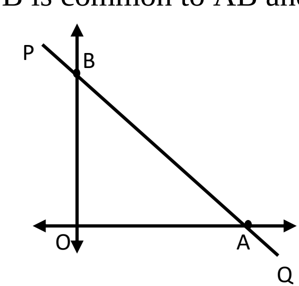
$$= \frac{8-4}{7-3} = 1$$

Slope of BC $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{12-8}{11-7}$

$= 1$

Hence, slope of AB and slope of BC are same (i.e. 1). B is common to AB and BC. So A, B, C are collinear points.

Intercepts in the axes

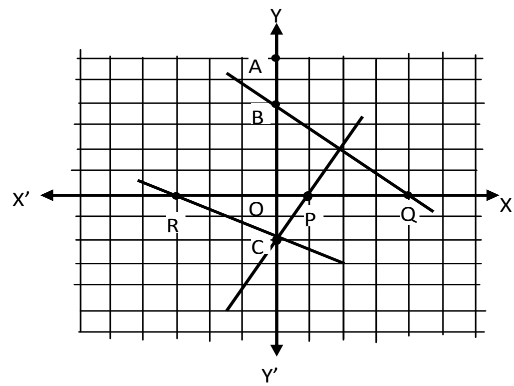


Suppose PQ be a line which intersects X-axis at A and Y-axis at B. The distance of point A from origin

O is called x-intercept and the distance of point B from origin O is called y-intercept. Generally, they are denoted by a and b respectively.

ie. OA = x-intercept (a)

OB = y-intercept (b)



Example 5:

Find x-intercept and y-intercept of the given lines in graph.

Solution: Here,

Three straight lines BQ, RC and PC are given.

Now, the intercepts of the straights line BQ, RC and PC are:

Line	X-intercept (a)	Y-intercept (b)
BQ	4	4
RC	-3	–2
PC	1	–2

Exercise 4.3

1. **Find slope of straight lines having following inclinations:**

   (a) 30° (b) 45° (c) 0° (d) 60°

2. **Find the inclination of straight lines with following slope.**

   (a) 1 (b) $\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) 0

3. **Find the slope of the line joining the following points.**

   (a) (2, 5) and (3, 4) (b) (1, 3) and (9, 1) (c) (6, 2) and (4, 3)

   (d) (–3, 5) and (5, 9) (e) (4, 3) and (7, 5)

4. (a) If slope of line joining (2, y) and (4, 5) is 1 find the value of y.

   (b) Find the value of x, if the slope of (4, 3) and (x, 5) is $\frac{2}{3}$

   (c) What will be the value of k if the slope of line joining (6, k) and (4, 3) is $\frac{-5}{2}$.

5. **Find the equation of following straight lines.**

(a) X-axis (b) Y-axis (c) 3 units right to Y–axis

(d) 2 units left to Y-axis (e) 5 units above X-axis

(f) 4 units below X-axis (g) Passing through (3, 2) parallel to X-axis

(h) Passing through (–3, –3) parallel to Y-axis

6. **Show the following points are collinear.**

(a) (2, 5), (5, 8), (8, 11) (b) (–2, 3), (2, 5) and (8, 8)

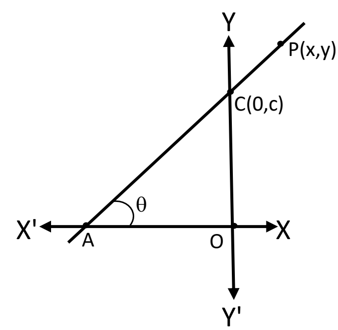
7. (a) Find intercepts of line joining (2, 4) and (5, 1) by plotting on Cartesian plane.

(b) Find intercepts of line joining (5, 5) and (8, 2) by plotting on Cartesian plane.

4.4 Equations of Straight Lines in Standard Forms

Slope intercept from

(When slope of a straight line m and y-intercept 'c' are given)



Let AB be a straight line which meets *x*-axis at A and y-axis at the point C. Let θ be the angle made by AB with x-axis in positive direction then ∠BAX = θ, OC = c then the coordinates of C is (0, c).
Now, tan θ = m is slope of AB.
Let P (x, y) be any point on AB then slope of the line AB is same as the slope of line joining C(0, c) and P (x, y) and is given by m $= \frac{y-c}{x-0}$

$or, \quad y - c = mx$

$or, \quad y = mx + c,$ is required equation of straight line AB.

Note:

If the line AB passes through origin (0, 0) then y-intercept c = 0 and the equation of straight line is $y = mx + 0$. i.e, $y = mx$

i.e. If constant term is absent in a linear equation then it passes through origin.

Example 1

Find the equation of a straight line making angle 45° with x-axis and makes y-intercept 3.

Solution:

We have, Angle of inclination (θ) = 45°,

y- intercept (c) = 3

Slope (m) = tanθ = tan 45° = 1

Now, the required equation of line is $y = mx + c$

$or, y = 1x + 3$

$or, y = x + 3$

or, $x - y + 3 = 0$

which is required equation of straight line.

Example 2

What will be the equation of straight line which makes the angle of 60° with positive x-axis and meet y-axis at (0, –7)

Solution:

We have the angle of inclination (θ) = 60°

Slope (m) = tanθ = tan60° = $\sqrt{3}$

y-intercept (c) = –7

Now the equation of the straight line is y = mx + c

or, $y = \sqrt{3}x - 7$

or, $\sqrt{3}x - y - 7 = 0$ is required equation.

Example 3

Find the equation of straight lines which makes 45° with positive X- axis and -45° with negative X- axis and both pass through (0, 4).

Solution:

Let, AC and AB be straight lines passing through (0, 4). The angle made by AC with positive X-axis is 45° and the angle made by AB with positive X-axis is 135° = –45°.

For AC

m = tan 45° = 1

$c = 4$

The required line is $y = mx + c$

$or, \ y = 1x + 4$

$or, \ y = x + 4$

$or, x - y + 4 = 0$

For line AB

$m = \tan 135° = -1$

c = 4

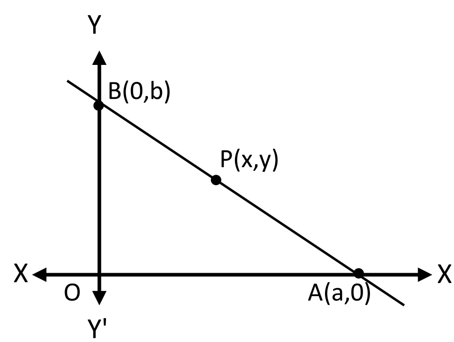
The required line is y = $mx+c$

or, $y = -1x + 4$

r, $x + y = 4$

$or,\ x + y - 4 = 0$ is required equation.

Double Intercept Form



The equation of straight line when two intercepts x-intercept (a) and y-intercept (b) are given.

Let AB be a straight line which intersects x-axis at A (a, 0) and y-axis at B (0, b). Let P (x, y) by any point on AB, then

Slope of the line joining A (a, 0) and B (0, b) is $m_1 = \frac{b-0}{0-a} = -\frac{b}{a}$

Also AP is the part of AB

Slope of AP = $m_2 = \frac{y-0}{x-a} = \frac{y}{x-a}$

Since AP is a part of AB, so slope of AP = slope of AB

or, $\frac{y}{x-a} = \frac{-b}{a}$

or, ay = –xb + ab

or, bx + ay = ab

Dividing both side by ab

$$\frac{bx}{ab} + \frac{ay}{ab} = \frac{ab}{ab}$$

Or, $\frac{x}{a} + \frac{y}{b} = 1$

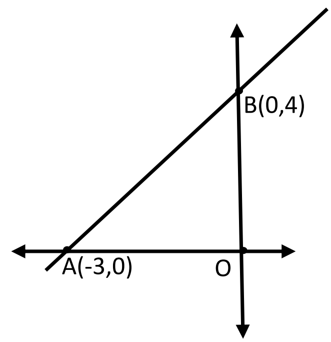
Since P(x, y) is arbitrary point on AB. So, $\frac{x}{a} + \frac{y}{b} = 1$ is the equation of AB.

Hence, $\frac{x}{a} + \frac{y}{b} = 1$ is required equation of straight line in double intercepts form.

Example 4

Find equation of a straight line which meets x-axis at –3 and y-axis at 4.

Solution:



We have,

X-intercept (a) = –3 Y- intercept (b) = 4

Now, equation of straight line is $\frac{x}{a} + \frac{y}{b} = 1$

Or, $\frac{x}{-3} + \frac{y}{4} = 1$

Or, $\frac{-4x + 3y}{12} = 1$

Or, $-4x + 3y = 12$

Or, $-4x + 3y - 12 = 0$

Or, $4x - 3y + 12 = 0$

is required equation of the straight line.

Example 5

Find the equation of a straight line which passes through (3, 5) and makes equal intercepts on x-axis and y-axis.

Solution:

Let the line makes both intercepts $a = b = k$.

$\frac{x}{k} + \frac{y}{k} = 1 \text{ or } x + y = k$………………… (i)

Since the line (i) passes through (3, 5) so, 3+5 = k

$k = 8$

Substituting the value of k in (i) we get

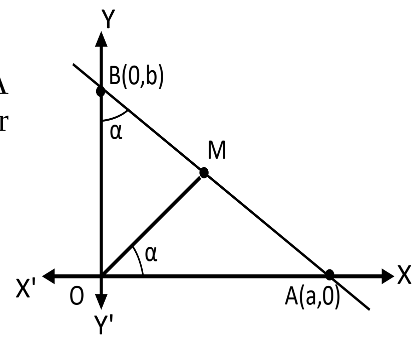
$x + y = 8$

Which is required equation.

(a) Perpendicular Form (Normal form) of Straight Line.

The equation of straight line when the length of perpendicular from origin to that line (p) and the angle made by this perpendicular with X-axis (α) is given.

Method I



Let AB be a straight line which meets x-axis at A and y-axis at B. Let OM be the perpendicular drawn from origin O to the line AB and OM = p.

Also $\angle MOA = \alpha$ and $\angle OBM = \alpha$.

So, $\angle BOM = 90° - \alpha$ and $\angle OBM = \alpha$.

Now, in right angled triangle OMA

$\cos\alpha = \frac{\text{base}}{hypotenus} = \frac{OM}{OA} = \frac{p}{OA}$

$\Rightarrow OA = \frac{p}{\cos\alpha}$

Similarly, in right angled triangle OMB

$\sin\alpha = \frac{OM}{OB} = \frac{p}{OB}$

or, $OB = \frac{p}{\sin\alpha}$

$\therefore$ The coordinates of A is $\left(\frac{p}{cos\alpha}, 0\right)$ and B is $\left(0, \frac{p}{sin\alpha}\right)$

Now, the equation of line in double intercept form is given by

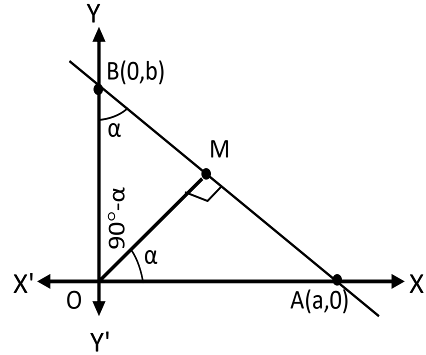
$$\frac{x}{a} + \frac{y}{b} = 1$$

or, $\dfrac{x}{\dfrac{p}{\cos\alpha}} + \dfrac{y}{\dfrac{p}{\sin\alpha}} = 1$

or, $\dfrac{x\cos\alpha + y\ \text{xin}\alpha}{p} = 1$

or, $x\cos\alpha + y\sin\alpha = p$ is required equation of straight line in normal form.

Method II



In figure, $\angle AOM = \alpha$

$\angle XAM = 90^0 + \alpha$

Slope of AB $= \tan(90 + \alpha)$

$= -\cot \alpha$

Again $\angle MOB = 90^0 - \alpha$ and $\angle OBM = \alpha$

In right angled triangle OMB.

$$\sin\alpha = \frac{OM}{OB}$$

$$\text{or, } OB = \frac{OM}{\sin\alpha} = \frac{p}{\sin\alpha}$$

$\therefore$ The equation of line with slope – cot α and y-intercept $\frac{p}{\sin\alpha}$ is

$$y = mx + c = -\cot\alpha x + \frac{p}{\sin\alpha}$$

$$\text{or, } y = \frac{-\cos\alpha}{\sin\alpha} x + \frac{p}{\sin\alpha}$$

$$\text{or, } y = \frac{-\cos\alpha x + p}{\sin\alpha}$$

$$\text{or, } y\sin\alpha = -x\cos\alpha + p$$

$$\text{or, } x\cos\alpha + y\sin\alpha = p$$

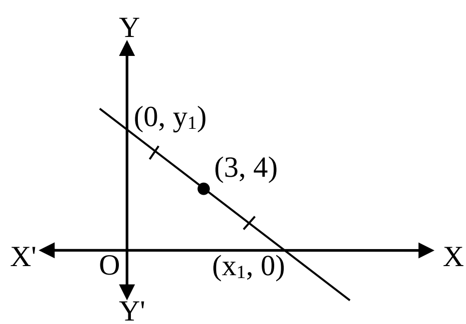
is required equation of straight line in normal form.

Example 6

Find the equation of the straight line in which the portion of which intercepted between the axes is bisected at (3,4).

Solution

We have, (3, 4) is midpoint of a line intercepted between the axes. Let $(x_1, 0)$ be a point in X-axis and $(0, y_1)$ be the point in Y-axis. Then, we have by using midpoint formula,



$$x = \frac{x_1 + x_2}{2}$$

$$\text{or, } 3 = \frac{x_1 + 0}{2}$$

$$\text{or, } x_1 = 6$$

Now, the equation of the straight line is passing with two points (3,4) and (6, 0) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or, $y - 4 = \frac{0 - 4}{6 - 3}(x - 3)$

or, $3y - 12 = -4x + 12$

or, $4x + 3y - 24 = 0$ is required equation of the straight line.

Example 7

Find the equation of a straight line if the length of perpendicular from origin to that line is $2\sqrt{2}$ units and the perpendicular is inclined with x-axis at 45°. Also show that it passes through (6, –2).

Solution:

We have, $\alpha = 45°$, $p = 2\sqrt{2}$ units.

The equation of straight line is

$$x\cos\alpha + y\sin\alpha = p$$

or, $x\cos 45° + y\sin 45° = 2\sqrt{2}$

or, $x\frac{1}{\sqrt{2}} + y\frac{1}{\sqrt{2}} = 2\sqrt{2}$

or, $(x + y)\frac{1}{\sqrt{2}} = 2\sqrt{2}$

or, $x + y = 2\sqrt{2} \times \sqrt{2} = 4$

$\therefore$ The required equation is $x + y = 4$. ....(i)

At (6, –2) the equation (i) becomes,

or, $x + y = 4 \Rightarrow 6 - 2 = 4$

or, $4 = 4$

Hence, the straight line $x + y = 4$ passes through (6, –2).

Exercise 4.4

1. **Find the equation of following straight lines.**
   (a) Slope (m) = 5 and y-intercept 3.
   (b) Inclination $\theta = 45°$ and y-intercept –4.

(c) Slope (m) $= \frac{2}{3}$ and y-intercept 6.

(d) Slope $m = \frac{1}{\sqrt{3}}$ and y-intercept (c) $= \sqrt{3}$

(e) Inclination 60° and y-intercept 0.

2. (a) What will be the equation of straight line which makes angle of 30° with positive x-axis and meets y-axis at (0, 4).

(b) What will be the equation of straight line which makes angle 120° with positive x-axis and meets y-axis at (0, –5).

(c) Find equation of straight line which makes angle 135° with x-axis and meets y-axis at (0, 6) Also show that it passes through (4, 2).

(d) Find the equation of straight line equally inclined to the axes and passing through (0, 4).

3. Find the equation of straight lines in following conditions.

(a) x-intercept = 4 y-intercept = –3

(b) x-intercept = –3 y-intercept = 3

(c) x-intercept = 5 y-intercept = 5

(d) x-intercept = –4 y-intercept = –3

4. (a) Find equation of straight line passing through the point (3, 5) and cutting off equal intercepts on x-axis and y-axis.

(b) Find the equation of straight line which passes through the point (6, 4) and has intercepts on the axis.

(i) Equal in magnitude and sign

(ii) equal in magnitude but opposite in sign

(c) Find the equation of straight line which passes through (3, 2) and making x-intercept double than y-intercept.

(d) Find the equation of the straight line of which passes through (–3, 8) and making intercepts on the axes whose product is 12.

5. (a) Find the equation of the straight line in which the portion of which intercepted by the axes is divided by the point (4, 1) in the ratio 1:2.

(b) Find the equation of straight line which passes through the point (–5, 6) and the portion of it between the axes is divided by the point in the ratio 3:4.

(c) Find the equation of straight line which passes through (–4, 3) such that the portion of it intercepted between axes is divided in the ratio 5:3 at that points.

6. Find the equation of straight line in following cases

(a) p = 2 units ,with $\alpha = 30°$ (b) p = 6 units, with $\alpha = 45°$

(b) p = 8 units, with $\alpha = 90°$ (d) p = 3 units, with $\alpha = 120°$

(c) p = 7 units, with $\alpha = 60°$ (f) p = $2\sqrt{2}$ units with $\alpha = 45°$

7. (a) What will be the equation of straight line in which p = $3\sqrt{2}$units and slope of p is 1. Show that it passes through (7, –1).

(b) What will be the equation of straight line with p = 3 and slope of p is $\frac{1}{\sqrt{3}}$.

(c) What will be the equation of straight line with length of perpendicular $\frac{5}{2}$ and slope of p is $\sqrt{3}$. Show that it passes through (–4, $\sqrt{3}$).

4.5 Reduction of the General Equation of straight line into Standard Form

Every straight line has its equation of the first degree in x and y. Conversely, we can say that every first degree equation in x and y represents a straight line.

The equation of the form Ax + By + C = 0 where A, B and C are constants and A and B cannot be simultaneously zero, is known as general equation of first degree in x and y.

Ax + By + C = 0 can be reduced into three standard form of straight lines.

(a) Reduction of Ax + By + C = 0 into slope – intercept form.

We have, the general equation of straight line is

Ax + By + C = 0

or, By = –Ax – C

Dividing on both sides by B

$\frac{By}{B} = \frac{-A}{B}x - \frac{C}{b} \quad or, y = \frac{-A}{B}x + \left(-\frac{C}{B}\right)$.............(i)

Comparing (i) with $y = mx + c$, we get

Slope (m) $= \frac{-A}{B}$ and y- intercept (c) $= \frac{-C}{B}$ in both cases B ≠ 0.

i.e. $m = \frac{-\,Coeff.\ of\ x}{Coeff.\ of\ y}\ and\ c = \frac{-\,Constant}{Coeff.\ of\ y}$

Example 1

Reduce x + 3y = 9 in slope intercept form.

Solution:

We have, $x + 3y = 9$

or $3y = -x + 9$

dividing both side by 3

$$\frac{3y}{3} = \frac{-1}{3}x + \frac{9}{3}$$

This gives $y = \frac{-1}{3}x + 3$ ....(i)

Comparing equation (i) with $y = mx + c$, we get

Slope (m) = $\frac{-1}{3}$ and y- intercept (c) = 3

(b) Reduction of Ax + By + C = 0 to double intercept form.

We have, the given general equation of straight line is

$$Ax + By + C = 0$$

or, $Ax + By = -C$

Dividing by –C on both sides, we get,

$$\frac{A}{-C}x + \frac{B}{-C}y = \frac{-C}{-C}$$

or, $\frac{x}{\frac{-c}{A}} + \frac{y}{\frac{-c}{B}} = 1$ ………… (i)

Comparing equation (i) with $\frac{x}{a} + \frac{y}{b} = 1$, we get

x-intercept (a) $= \frac{-C}{A} = \frac{-\,Constant}{Coeff.\ of\ x}$

$y-$ intercept (b) $= \frac{-C}{B} = \frac{-\,Constant}{Coeff.\ of\ y}$

Example 2

Reduce 4x + 3y – 12 = 0 into double intercept form.

Solution: Here,

The general form of equation is $4x + 3y - 12 = 0$

or, $4x + 3y = 12$

Dividing on both sides by 12, we get

$$\frac{4x}{12} + \frac{3y}{12} = \frac{12}{12}$$

or, $\frac{x}{3} + \frac{y}{4} = 1$………… (i)

Comparing equation (i) with $\frac{x}{a} + \frac{y}{b} = 1$

We get, x-intercept (a) = 3
y-intercept (b) = 4

Example 3

Find the intercepts on the axes made by the line having equation.
$\sqrt{3}x + 2y - 6 = 0$

Solution: Here,

The general form of equation is

$$\sqrt{3}x + 2y - 6 = 0$$

or, $\sqrt{3}x + 2y = 6$

Dividing on both sides by 6, we get

$\frac{\sqrt{3}x}{6} + \frac{2}{6}y = \frac{6}{6}$

or, $\frac{x}{\frac{6}{\sqrt{3}}} + \frac{y}{3} = 1$

or, $\frac{x}{\frac{2\times 3}{\sqrt{3}}} + \frac{y}{3} = 1$

or, $\frac{x}{2\sqrt{3}} + \frac{y}{3} = 1$ ………………. (i)

Comparing equation (i) with $\frac{x}{a} + \frac{y}{b} = 1$, we get

x-intercept (a) = $2\sqrt{3}$, y-intercept (b) = 3.

(c) Reduction of A*x* + B*y* + C = 0 into normal form

We have the general equation of a line is

Ax + By + C = 0 ………… (i)

The equation of a line in normal form is

$x\cos\alpha + y\sin\alpha = p$ ………….. (ii)

Equation (i) and (ii) will be identical if

$$\frac{A}{\cos\alpha} = \frac{B}{\sin\alpha} = \frac{-C}{p} = k$$

$\Rightarrow$ A = k$\cos\alpha$ …………….. (iii)
B = k$\sin\alpha$ ……………. (iv)
C = –pk ………………... (v)

from (iii) and (iv)
$A^2 + B^2 = k^2(\cos^2\alpha + \sin^2\alpha)$
or, $k^2 = A^2 + B^2$
or, $k = \pm\sqrt{A^2 + B^2}$

Putting the value of k in (v)

C = –pk

or, $p = -\frac{C}{k} = \pm\left(\frac{C}{\sqrt{A^2+B^2}}\right)$

$\therefore \cos\alpha = \frac{A}{k} = \pm\frac{A}{\sqrt{A^2+B^2}}, \ \sin\alpha = \frac{B}{k} = \pm\frac{B}{\sqrt{A^2+B^2}}$

$\therefore$ The equation of a straight line in the normal form is

$$\pm\frac{A}{\sqrt{A^2+B^2}}x \pm \frac{B}{\sqrt{A^2+B^2}}y = \pm\frac{C}{\sqrt{A^2+B^2}}$$

As the perpendicular distance (p) is always positive, therefore c has to be selected positive or negative to make p always positive.

Steps for reducing in the normal form:

1. Divide on both sides of given equation by $\sqrt{(coeff.of\ x)^2 + (coeff.of\ y)^2}$.
2. Make the constant term in RHS positive.
3. Compare the equation obtained in 2 with $x\cos\alpha$ + y$\sin\alpha$ = p and find the value of α and p.

Example 4

Reduce $x + \sqrt{3y} + 4 = 0$ in to normal form.

Solution

We have the general form of line $x + \sqrt{3}y - 4 = 0$

Comparing with Ax + By + C = 0, we get

Where A = 1, B = $\sqrt{3}$ and C = 4

Now, $\sqrt{A^2 + B^2} = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4} = 2$

So, $\frac{x}{2} + \frac{\sqrt{3}}{2} y = \frac{4}{2}$

or, $\frac{1}{2}x + \frac{\sqrt{3}}{2} y = 2$........................(i)

Comparing (i) with $x \cos \alpha + y \sin \alpha = p$

$\cos \alpha = \frac{1}{2}$, $\sin \alpha = \frac{\sqrt{3}}{2}$ and p = 2

or, $\cos \alpha = \cos 60^0$, $\sin \alpha = \sin 60°$

∴ The required equation is $x\cos 60° + y \sin 60° = 2$

Example 5

Reduce $\sqrt{3}x - y + 2 = 0$ in three standard forms.

Solution:

Here, the equation of line is

$\sqrt{3}x - y + 2 = 0 \quad \dots \dots (i)$

For slope intercept form

Here, $\sqrt{3}x - y + 2 = 0$

or, $y = \sqrt{3}x + 2$ ...... (ii)

Comparing (ii) with $y = mx+c$

m = $\sqrt{3}$ and c = 2

For double intercept form

We have, $\sqrt{3}x - y + 2 = 0$

or, $\sqrt{3}x - y = -2$

Dividing both side of (i) by – 2

$$\frac{\sqrt{3}}{-2}x - \frac{y}{-2} = \frac{-2}{-2}$$

or, $\frac{x}{\frac{-2}{\sqrt{3}}} + \frac{y}{2} = 1$ .... (iii)

Comparing (ii) with $\frac{x}{a} + \frac{y}{b} = 1$ we get

x – intercept (a) = $\frac{-2}{\sqrt{3}}$, $y -$ intercept (b) = 2

For normal form

We have, $\sqrt{3}x - y + 2 = 0$

or, $\sqrt{3}x - y = -2$ ………… (iv)

Now, $\sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = \sqrt{3+1} = \sqrt{4} = 2$

Dividing both sides of equation (iv) by 2, we get

$$\frac{\sqrt{3}}{2}x - \frac{1}{2}y = \frac{-2}{2}$$

or, $-\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 1$.......... (v)

Comparing (v) with $x\cos\alpha + y\sin\alpha = p$

$\cos\alpha = \frac{-\sqrt{3}}{2} = \cos 150°$, $\sin\alpha = \sin 150° = \frac{1}{2}$, and p = 1

Hence, the required equations in three standard forms are

$y = \sqrt{3}x + 2,$

$\frac{x}{\frac{2}{\sqrt{3}}} + \frac{y}{2} = 1$ and

$x\cos150° + y\sin150° = 1$

Exercise 4.5

1. Reduce the following equations into slope intercept form. Also find slope and y-intercept.

(a) $4x + y + 3 = 0$ (b) $3y - 8x + 6 = 0$ (c) $5x + 3y - 9 = 0$

(d) $12x - 3y + 5 = 0$ (e) $6x + 2y + 3 = 0$

2. Reduce the following equations into double intercept form and find x-intercept and y-intercept.

(a) $3x - 4y - 12$ (b) $x - y + 3 = 0$ (c) $x + y - 5 = 0$

(d) $x - y + 4 = 0$ (e) $2x - 5y - 10 = 0$ (f) $3x - y + 27 = 0$

3. Reduce the following equation into normal form and hence find the value of p and α.

(a) $x + \sqrt{3}y = 4$ (b) $x + y = 2$ (c) $\sqrt{3}x + 2y = 11$

(d) $3x + 4y + 25 = 0$ (e) $y = \sqrt{3}x + 6$

4. Reduce the following equation into slope intercept form, double intercept form and normal form.

(a) $\sqrt{3}x + y + 6 = 0$ (b) $2\sqrt{3}x + 2y = 11$

(c) $4x + 6y - 3\sqrt{2} = 0$ (d) $x - \sqrt{3}y - 6 = 0$

5. (a) The equation $3x + 4y = 12$ meets the x-axis and y-axis at A and B. Find the area of right angled triangle OAB.

(b) A straight line $4x + 7y = 28$ meets x-axis at A and y-axis at B What will be the area of triangle OAB?

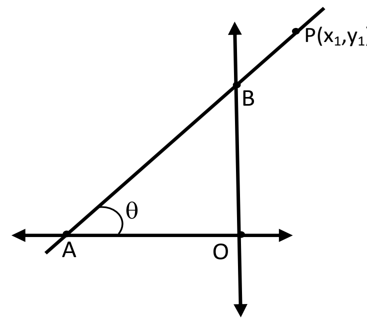
6. If the area of a right angled triangle AOB with $\angle AOB = 90^0$ is 16 sq. unit, find possible coordinates of A and B.

4.6 Other forms of Equations of straight Line

(a) Equation of a straight line in point slope form

"To find the equation of a straight line passing through a given point (x_1, y_1) and that makes given angle of inclination θ with x-axis"

Let P (x_1, y_1) be any point on the straight line AB which makes an angle of inclination θ with x-axis.



So slope (m) = tanθ

Suppose y-intercept is c then equation of straight line AB is

y = mx + c ……. (i)

Since line (i) passes through Q(x_1, y_1)

So, $y_1 = mx_1 + c$………………. (ii)

By subtracting (ii) from (i), we get

$y - y_1 = m(x - x_1)$.

The equation of a straight line with slopes 'm' and passes through (x_1, y_1) is **$(y - y_1) = m(x - x_1)$.**

Example 1

Find equation of a straight line passing through (3, –2) and makes an angle 45° with positive X-axis.

Solution:

We have (x_1, y_1) = (3, –2)

Angle (θ) = 45^0

Slope (m) = tanθ = tan45° = 1

∴ The required equation of straight line is

$y - y_1 = m(x - x_1)$

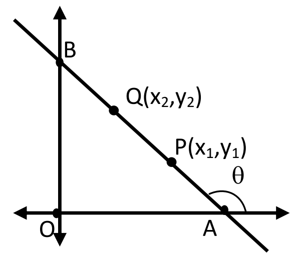
or, $y - (-2) = 1(x - 3)$

or, $y + 2 = x - 3$

or, $x - y - 5 = 0$ is the required equation.

(b) Equation of a line in two points form

"To find the equation of line passing through given two points (x_1, y_1) and (x_2, y_2)."



Let the angle of inclination of a line AB is θ

Which makes y-intercept 'c'.

Then its equation is $y = mx + c$ ----------- (i)

Let P (x_1, y_1) and Q (x_2, y_2) be two points, then the equation of straight line in slope intercept form become

$y_1 = mx_1 + c$ ----------- (ii)

$y_2 = mx_2 + c$ ------------- (iii)

By subtracting (ii) from (iii), we get

$y_2 - y_1 = m(x_2 - x_1)$

or, $m = \frac{y_2 - y_1}{x_2 - x_1}$ ------------ (iii)

Also, subtracting (ii) from (i), we get

$y - y_1 = m(x - x_1)$ ------------ (iv)

Putting value of m from (iii) we get

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

The required equation of straight line passing through given two points (x_1, y_1) and (x_2, y_2) is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Example 2

Find equation of line passes through (–3, 4) and (3, 6).

Solution:

We have (x_1, y_1) = (–3, 4)

(x_2, y_2) = (3, 6)

The equation of line through given two points is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or, $y - 4 = \frac{6-4}{3+3}(x+3)$

or, $y - 4 = \frac{2}{6}(x+3)$

or, $y - 4 = \frac{1}{3}(x+3)$

or, $3y - 12 = x + 3$

or, $x - 3y + 15 = 0$ is required equation of straight line.

Example 3

Show that three points (1, 3), (2, 4) and (3, 5) are collinear points.

Solution:

We have the three points are (1, 3), (2, 4) and (3, 5)

The equation of line through (1, 3) and (2, 4) is

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1)$$

or, $y - 3 = \frac{4-3}{2-1}(x-1)$

or, $y - 3 = \frac{1}{1}(x-1)$

or, $y - 3 = x - 1$

or, $x - y + 2 = 0$------(i)

If (i) passes through (3, 5) then

$x - y + 2 = 0$

or, $3 - 5 + 2 = 0$

or, $5 - 5 = 0$

or, $0 = 0$ (true)

Hence, the given three points are collinear.

> To verify the given three points are collinear, first find a straight line from any two points and then satisfy it by substituting the value of remaining point.

Example 4

Find the equation of straight line passing through (5, 5) and bisects the line joining (–3, 5) and (–5, –1).

Solution: Here,

The midpoint of line joining (–3, 5) and (–5, –1) is

$$x = \frac{x_1+x_2}{2}, \quad y = \frac{y_1+y_2}{2}$$

$x = \frac{-3-5}{2}$, $y = \frac{5-1}{2}$

$x = -4$ $y = 2$

∴ The midpoint of the straight line joining (–3, 5) and (–5, –1) is (–4, 2)

Now, the equation of straight line passing through (–4, 2) and (5, 5) is

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$y - 2 = \frac{5-2}{5+4}(x+4)$

or, $y - 2 = \frac{3}{9}(x+4)$

or, $y - 2 = \frac{1}{3}(x+4)$

or, $3y - 6 = x + 4$

or, $x - 3y + 10 = 0$

∴ The required equation of the straight line is $x - 3y + 10 = 0$

Example 5

Find equation of median from A (2, 2) of a triangle having coordinates of vertices A (2, 2), B (2, 8), and C (-6, 2).

Solution:

Let D be the midpoint of BC. Then the coordinates of D is

$$\left(\frac{-6+2}{2}, \frac{2+8}{2}\right) = (-2, 5)$$

Again the equation of straight line passing through A(2, 2) and D(–2, 5) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or, $y - 2 = \frac{5-2}{-2-2}(x-2)$

or, $y - 2 = -\frac{3}{4}(x-2)$

or, $4y - 8 = -3x + 6$

or, $3x + 4y - 14 = 0$

∴ Equation of median from A(2, 2) of a triangle having coordinates of vertices A(2, 2), B(2, 8) and C(–6, 2) is $3x + 4y - 14 = 0$.

Note: *Height drawn from vertex to the base (unequal side) in an isosceles triangle and height of equilateral triangle are also medians.*

Exercise 4.6

1. **Find the equation of straight line in following cases**.
   (a) Passing through (3, 5) and makes angle of 60^0 with x-axis.
   (b) Passing through (–2, –4) and makes angle of 150^0 with x-axis.
   (c) Passing through (–5, 2) with angle 120^0 with x-axis.
   (d) Passing through (6, –5) and makes angle 45^0 with positive x-axis.
   (e) Making angle of 30^0 and passes through (7, 4).
2. **Find the equation of straight line in following cases.**
   (a) Passing through (3, 5) and (–4, 3).
   (b) Passing through (–5, 6) and (–4, 5).
   (c) Passing through (–2, –7) and (3, –4).
   (d) Passing through (4, 8) and (–4, –8).
   (e) Passing through (a, 0) and (0, b).
3. **Show that the following points are collinear.**
   (a) (1, 9), (4, 10), (7, 11) (b) (–1, 3), (1, –1), (2, –3)
   (c) (5, 6), (3, 4) and (8, 9) (d) (3, 2), (5, 0), (8, –3)
   (e) (a, 0), (0, b) and (3a, –2b) (f) (3, 6), (–3, 4) and (6, 7)
4. (a) Find the equation of a straight line bisecting the line joining (3, 4) and (5, 6) and having an angle of inclination 135°.
   (b) Prove that the line joining (3, 5) and (–2, 7) bisects the line joining the point (7, 21) and (9, 4).
   (c) Find equation of line passing through origin and bisects the line joining the points (1, –2) and (4, 3).
   (d) Find the equation of the straight line that passes through origin and midpoint of the portion of line $3x+y = 12$ intercepted between axes.
   (e) Prove that the line joining (1, 2) and (2, –2) bisects the line joining (–3, 6) and (5, –2).
5. (a) Find the equation of each sides of a triangle whose vertices are (–1, 3), (1, –1) and (5, –2).

(b) Find equation and length of median of triangle ABC through A (2, 3) having coordinates of vertices A (2, 3), B (4, 5) and C (6, –3).

(c) If the point P (a, b) lines in $x–2y – 3 = 0$ and Q (b, a) lies in $2x+y–11 = 0$ then find equation of length of PQ.

(d) If A (8, 0), B (1, 8) and C (5, –2) be coordinates of three vertices of a triangle ABC, Prove that the equation of median from (5, –2) is $x+7y –4 = 0$.

6. (a) If (3, 6), (–3, 4) and (a, 7) are collinear find value of a.

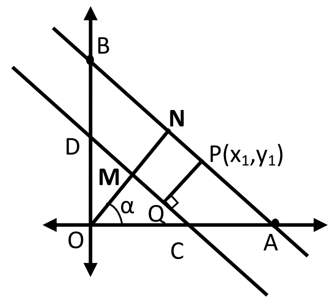
(b) If (8, –3), (5, 0) and (3, b) are collinear, then find the value of b.

(c) If (6, k), (10, 8) and (14, 10) are collinear then find value of k.

(d) If (a, p), (0, b) and (3a, -2b) are collinear what will be the value of p.

4.7 Distance between a Point and a Straight Line

(a) *Perpendicular distance (length of perpendicular) from a point to* $x\cos\alpha + y\sin\alpha = p$



Suppose CD is a straight line with equation

$x\cos\alpha + y\sin\alpha = p$ ………… (i)

So the length of perpendicular from origin O(0, 0) to CD is p = OM and ∠MOC= α

Let P (x_1, y_1) be a point on AB from which PQ ⊥ CD is drawn then AB || CD ⇒ MN ⊥ AB and ON = P_1.

The equation of line AB is

$x_1\cos\alpha + y_1\sin\alpha = p_1$ ……………. (ii)

Since P(x_1, y_1) lies on (ii)

Now, PQ = MN = ON – OM $= p_1 - p = x_1\cos\alpha + y_1\sin\alpha - p$.

The perpendicular distance (Length of perpendicular) drawn from point (x_1, y_1) to the straight line $x\cos\alpha + y\sin\alpha = p$ is $x_1\cos\alpha + y_1\sin\alpha - p_1$.

Example 1

Find the length of perpendicular from point (–1, 3) to $x - y + 2\sqrt{2} = 0$

Solution

We have the given line is

$x - y + 2\sqrt{2} = 0$ ………. (i)

Reducing (i) into normal form

$$\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} + \frac{2\sqrt{2}}{\sqrt{2}} = 0$$

⇒ $\cos 45° \times x - \sin 45° \times y + 2 = 0$

Now the perpendicular distance from (–1, 3) to $x\cos 45° - y\sin 45° + 2 = 0$ is

$$d = -1 \times \frac{1}{\sqrt{2}} - 3 \times \frac{1}{\sqrt{2}} + 2$$

$$= \frac{-1}{\sqrt{2}} - \frac{3}{\sqrt{2}} + 2$$

$$= \frac{-1-3+2\sqrt{2}}{2} = \frac{-4+2\sqrt{2}}{2} = -\frac{2(2-\sqrt{2})}{2} = -(2-\sqrt{2})$$

Since d is always positive.

Therefore, length of perpendicular from point (–1, 3) to $x - y + 2\sqrt{2} = 0$ is $2 - \sqrt{2}$.

(b) The length of perpendicular from (x_1, y_1) to *Ax* + *By* + *C* = *0*.

Let the given line is *Ax* + *By* + *C* = *0* .... (i)

Reducing (i) to normal form, we get

$$\frac{Ax}{\sqrt{A^2+B^2}} + \frac{By}{\sqrt{A^2+B^2}} + \frac{C}{\sqrt{A^2+B^2}} = 0$$

If C is negative then,

$$\frac{Ax}{\sqrt{A^2+B^2}} + \frac{By}{\sqrt{A^2+B^2}} = \frac{C}{\sqrt{A^2+B^2}}$$

Where $\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}, \quad sin\alpha = \frac{B}{\sqrt{A^2+B^2}}, \quad p = \frac{C}{\sqrt{A^2+B^2}}$

If C is positive, $\frac{Ax}{\sqrt{A^2+B^2}} + \frac{By}{\sqrt{A^2+B^2}} = \frac{C}{\sqrt{A^2+B^2}}$

Where $\cos\alpha = \frac{A}{\sqrt{A^2+B^2}}, sin\ \alpha = \frac{B}{\sqrt{A^2+B^2}},\ p = \frac{-C}{\sqrt{A^2+B^2}}$

The length of perpendicular from P (x_1, y_1) to *Ax* + *By* + *C* = *0* is

$p = \pm|x_1 cos\alpha + y_1 \sin\alpha - p|$ [∵ p is perpendicular distance, so we take absolute value]

$$= \pm\left[x_1 \times \frac{A}{\sqrt{A^2+B^2}} + y_1 \times \frac{B}{\sqrt{A^2+B^2}} + \frac{C}{\sqrt{A^2+B^2}}\right]$$

$$= \left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|$$

The length of perpendicular (P) $= \left\|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right\|$

Example 2

Find length of perpendicular from (–3, –4) to the line 3x + 4y – 7 = 0

Solution

Here, 3*x* + 4*y* – *7* = *0* and (x_1, y_1) = (–3, –4)

The length of perpendicular is

$$p = \left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|$$

$= \left|\frac{3\times(-3)+4\times(-4)-7}{\sqrt{3^2+4^2}}\right|$

$= \left|\frac{-9-16-7}{\sqrt{25}}\right| = \left|\frac{-32}{5}\right| = \frac{32}{5}$

Therefore the length of perpendicular (p) = $\frac{32}{5}$ units

Example 3

Find the distance between the following pair of parallel lines 2x + y = 5 and 2x + y = 3.

Solution

We have, $2x + y = 5$ .... (i)

$2x + y - 3 = 0$.............. (ii)

Distance of line (i) from origin is (d_1) = $\left|\frac{-5}{\sqrt{2^2+1^2}}\right| = \frac{5}{\sqrt{5}}\ units$

Distance of line (ii) from origin is (d_2) = $\left|\frac{-3}{\sqrt{2^2+1^2}}\right| = \frac{3}{\sqrt{5}}\ units$

Distance between parallel lines is

$d_1 - d_2 = \frac{5}{\sqrt{5}} - \frac{3}{\sqrt{5}}$ units $= \frac{2}{\sqrt{5}}$ units

Alternatively, putting x = 0 in (i) we get, y = 5

Therefore, (0, 5) lies in line (i).

∴ The distance between (0, 5) and $2x + y - 3 = 0$ is

$\left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|$

$= \left|\frac{2.0+5-3}{\sqrt{2^2+1^2}}\right|$

$= \left|\frac{2}{\sqrt{5}}\right|$

$= \frac{2}{\sqrt{5}}$ units

Therefore the distance from (0, 5) to $2x + y - 3 = 0$ is $\frac{2}{\sqrt{5}}$ units

Exercise 4.7

1. Find the perpendicular distance of a point and a line given below.

   (a) $x\cos30° + y\sin30° - 3 = 0$ from (3, 2)

   (b) $x\cos45° + y\sin45° + 5 = 0$ from (3, 4)

(c) $x\cos 60° + y\sin 60° + 8 = 0$ from origin.

(d) $3x + 4y + 5 = 0$ from point (0, 0)

(e) $\sqrt{3}x - y = 20$ form $(2\sqrt{3}, 4)$

(f) $px + qy = q^2 + p^2$ from (0, 0)

(g) $x + y = 10$ from (3, 6)

(h) $3x - 4y + 15 = 0$ from (2, 1)

2. Find the distance between following pair of parallel lines.

(a) $x + y + 3\sqrt{2} = 0$ and $2x + 2y - 10\sqrt{2} = 0$

(b) $3x + 5y = 6$ and $3x+5y+23 = 0$

(c) $2x + 3y = 6$ and $4x+6y+7 = 0$

(d) $x + y - 5\sqrt{2} = 0$ and $2x + 2y + 6\sqrt{2} = 0$

(e) $3x+2y+4 = 0$ and $3x + 4y - 16 = 0$

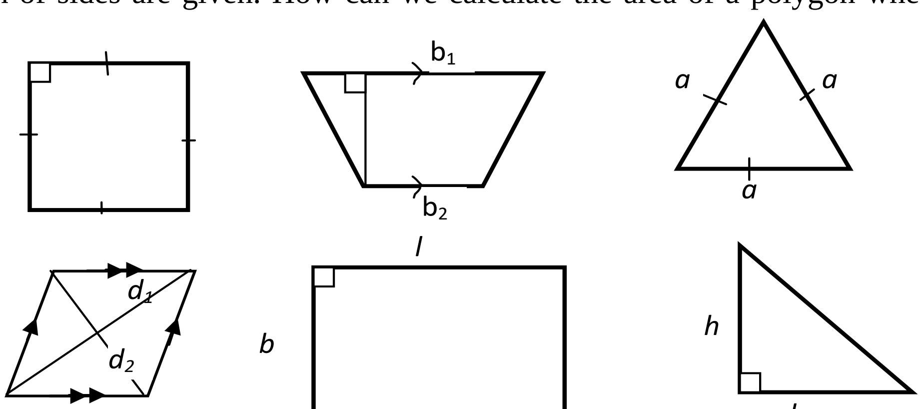
3. (a) What will be the value of k if the distance from (2, –3) to the line $kx-4y+7 = 0$ is 5 units.

(b) If the length of perpendicular from (–2, y) to line $4x-3y+10 = 0$ is 4 units find the value of y.

4.8 Area of Triangle and Quadrilateral

Discuss in small groups and identify the formulae of area of the following plane figures. Also, discuss if there are any other techniques and methods of calculating the area of the shapes?

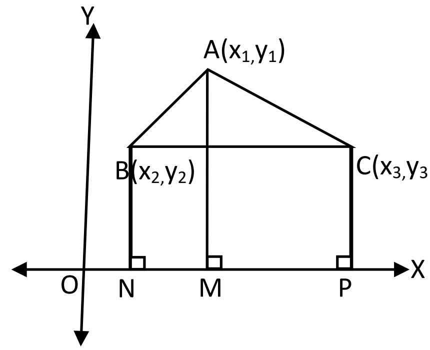
We can easily find the area of triangle and quadrilaterals when their sides and length of sides are given. How can we calculate the area of a polygon when the



coordinates of vertices are given? Let us discuss about it.

Area of triangle

Let A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) be the vertices of a triangle ABC. Draw AM, BN and CP perpendiculars on OX such that AM = y_1, BN = y_2 and CP = y_3, OM = x_1, ON = x_2, OP = x_3.



From figure,

Area of ΔABC = Ar(trap. ABNM) + Ar(trap. AMPC) – Ar (trap. BNPC)

We know that area of trapezium $= \frac{1}{2}$ (sum of parallel bases) × height

$= \frac{1}{2}$ (sum of parallel bases × Perpendicular distance between them)

So, Ar (ΔABC) $= \frac{1}{2}$ (BN+AM)× NM+$\frac{1}{2}$ (MA+CP)×MP – $\frac{1}{2}$ (BN+CP)×NP

$\Delta = \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3) \times (x_3 - x_2)$

$\Delta = \frac{1}{2}(x_1y_2 - x_2y_2 + x_1y_1 - x_2y_1 + x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3 - x_3y_2 + x_2y_2 - x_3y_3 + x_2y_3)$

$= \frac{1}{2}(x_1y_2 - x_2y_1 + x_3y_1 - x_1y_3 - x_3y_2 + x_2y_3)$

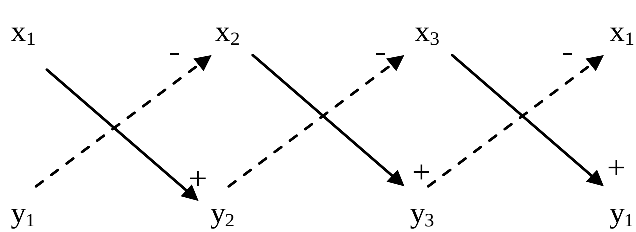
$= \frac{1}{2}(x_1y_2 - x_1y_3 + x_2y_3 - x_2y_1 + x_3y_1 - x_3y_2)$

$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$ sq. unit

Hence the area of ΔABC with vertices A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) is denoted by Δ and given by,

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \text{ sq. units.}$$

We can put the expression $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = (x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$ as,



$\therefore\ \Delta = \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$

Note: *If the area of triangle formed by three coordinates is zero, then these three points are collinear points.*

Example 1

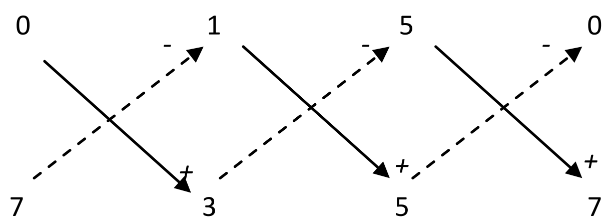
What will be the area of a triangle with three vertices A (0, 7), B (1, 3) and C (5, 5)?

Solution: We have, $x_1 = 0,\ y_1 = 7$

$x_2 = 1,\ y_2 = 3$

$x_3 = 5,\ y_3 = 5$

We can arrange these coordinates as



The area of ΔABC $= \frac{1}{2}(0 \times 3 - 7 \times 1 + 1 \times 5 - 3 \times 5 + 5 \times 7 - 5 \times 0$

$= \frac{1}{2}(0 - 7 + 5 - 15 + 35 - 0)$

$= \frac{1}{2} \times 18$

= 9 square units.

Alternatively, by using formula

Ar(ΔABC) $= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$ sq units.

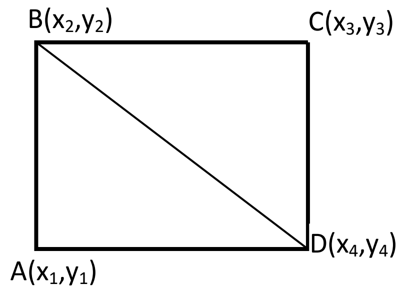
$= \frac{1}{2}[0(3 - 5) + 1(5 - 7) + 5(7 - 3)]$

$= \frac{1}{2}[0 + 1 \times (-2) + 5 \times 4] = \frac{1}{2}(-2 + 20)$

$= \frac{1}{2} \times 18$

= 9 square units.

Area of Quadrilateral



Let us consider A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) and D(x_4, y_4) be coordinates of four vertices of a quadrilateral ABCD.

Draw diagonal BD. Then the area of quadrilateral ABCD is equal to sum of area of Δ ABD and area of ΔBCD.

∴ The area of ABCD = Ar(ΔABD) + Ar(ΔBCD)

Ar(ΔABD) $= \frac{1}{2} x_1(y_2 - y_4) + x_2(y_4 - y_1) + x_4(y_1 - y_2)$

Ar(ΔABD) $= \frac{1}{2} x_2(y_3 - y_4) + x_3(y_4 - y_2) + x_4(y_2 - y_3)$

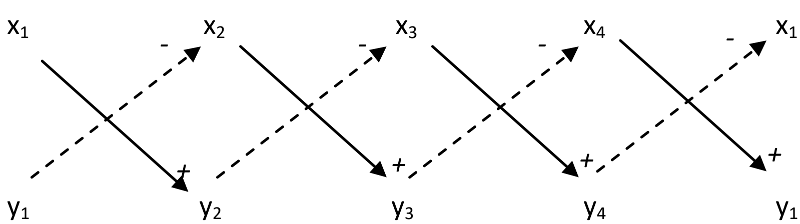
Now, Area of ABCD

$= \frac{1}{2}[x_1(y_2 - y_4) + x_2(y_4 - y_1) + x_4(y_1 - y_2)] + \frac{1}{2}[x_2(y_3 - y_4) + x_3(y_4 - y_2) + x_4(y_2 - y_3)]$

$= \frac{1}{2}[x_1y_2 - x_1y_4 + x_2y_4 - x_2y_1 + x_4y_1 - x_4y_2 + x_2y_3 - x_2y_4 + x_3y_4 - x_3y_2 + x_4y_2 - x_4y_3]$

$= \frac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4]$ sq. units

We can express the expression inside the bracket as



$[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4]$

∴ Area of quadrilateral $= \frac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + x_4y_1 - x_1y_4]$

Example 2

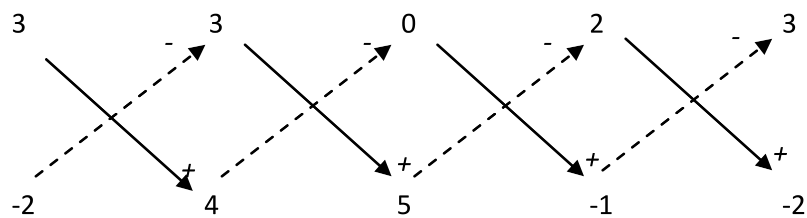
Find area of quadrilateral formed from the given four coordinates A(3, –2), B(3, 4), C(0, 5) and D(2, –1).

Solution

We have, $x_1 = 3, \quad x_2 = 3, \quad x_3 = 0, \quad x_4 = 2$

$y_1 = -2, \quad y_2 = 4, \quad y_3 = 5, \quad y_4 = -1$

So by arranging the coordinates as



∴ Area of ABCD

$= \frac{1}{2}[x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_4 - x_4 y_3 + x_4 y_1 - x_1 y_4]$ sq. units

$= \frac{1}{2}[3 \times 4 - 3 \times (-2) + 3 \times 5 - 4 \times 0 + 0 \times (-1) - 5 \times 2 + 2 \times (-2) - (-1) \times 3]$

$= \frac{1}{2}(12 + 6 + 15 - 10 - 4 + 3) = \frac{1}{2}(22) = 11$ sq. units.

∴ Area of quadrilateral ABCD = 11 sq. units.

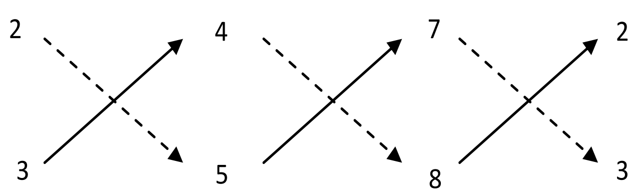
Example 3

Show that (2, 3), (4, 5) and (7, 8) are collinear.

Solution:

We have the area of triangle formed by three points is calculated as

Now, the area of triangle



Area of triangle = 0 means the points are collinear.

$= \frac{1}{2}[2 \times 5 - 3 \times 4 + 4 \times 8 - 5 \times 7 + 7 \times 3 - 8 \times 2]$ sq. units

$= \frac{1}{2}(10 - 12 + 32 - 35 + 21 - 16)$ sq. units

$= \frac{1}{2}(63 - 63) = 0$ sq.units

Since the triangle formed by three points have area zero so these point are collinear.

Example 4

If (a, 0), (x, y) and (0, b) are collinear points then prove that $\frac{x}{a} + \frac{y}{b} = 1$

Solution

We have, $x_1 = a \quad x_2 = x, \quad x_3 = 0$

$y_1 = 0, \quad y_2 = y, \quad y_3 = b$

Since three points are collinear. So area of triangle made by three points is zero

So, $\frac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3] = 0$

or, $\frac{1}{2}(a \times y - x \times 0 + x \times b - 0 \times 0 - a \times b) = 0$

or, $ay + bx - ab = 0$

or, $bx + ay = ab$

or, $\frac{bx}{ab} + \frac{ay}{ab} = \frac{ab}{ab}$

or, $\frac{x}{a} + \frac{y}{b} = 1$

Example 5

If A(3, 1), B (11, 1) and C (8, 6) respectively be three coordinates of ΔABC. If D is a point (x, y) then prove that ΔADB:ΔABC = (y – 1) : 5.

Solution:

We have, $x_1 = 3, \quad x_2 = 11, \quad x_3 = 8 \quad x_4 = x$

$y_1 = 1, \quad y_2 = 1, \quad y_3 = 6 \quad y_4 = y$

With A (3, 1), B (11, 1), C (8, 6), D (x, y)

Area of ΔABC $= \frac{1}{2}[x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1]$ sq.units

$= \frac{1}{2}(3 \times 1 + 11 \times 6 + 8 \times 1 - 1 \times 11 - 1 \times 8 - 6 \times 3)$

$= \frac{1}{2}(3 + 66 + 8 - 11 - 8 - 18)$ sq.units

$= \frac{1}{2} \times 40 = 20$ sq.units

Area of ΔABD $= \frac{1}{2}[x_1y_2 + x_2y_4 + x_4y_1 - y_1x_2 - y_2x_4 - y_4x_1]$

$= \frac{1}{2}(3 \times 1 + 11 \times y + x \times 1 - 1 \times 11 - 1 \times x - y \times 3)$

$= \frac{1}{2}(3 + 11y + x - 11 - x - 3y)$

$= \frac{1}{2}(8y - 8) = 4(y - 1)$sq.units

Now, $\frac{\text{Ar}(\Delta ABD)}{\text{Ar}(\Delta ABC)} = \frac{4(y-1)}{20} = \frac{y - 1}{5}$

∴ ΔABD : ΔABC = (y – 1) : 5 Proved.

Exercise 4.8

1. Find the area of triangle formed by following coordinates.

(a) P(3, 2), Q(–2, 5), R(2, –3)
(b) A(1, 0), B(0, 2), C(–1, 2)
(c) A(–2, 5), B(3, 1), C(2, 5)
(d) X(2, 2), Y(6, 2), Z(4, 4)
(e) P(0, 2), Q(2, 0), R(5, 2)
(f) A(4, 0), B (0, 0), C(0, 5)
(g) K(2, 6), A(3, 8), R (–1, 0)
(h) A (6, 3), T(–3, 5), R (4, –2)
(i) (3, –5), (–2, –7), (18, 1)
(j) C (4, 6), B(0, 4), Z(6, 2)

2. Show that the following points are collinear.

(a) P (3, 1), Q(5, 4) and R(2, 2)
(b) K (2, 3), I(4, 5) and C(7, 8)
(c) K (3, –2), L (1, 3) and M(4, 0)
(d) A(–5, 1), B(5, 5) and C(10, 7)
(e) $\left(0, \frac{3}{2}\right)\left(2, \frac{1}{2}\right)\left(\frac{-1}{2}, 2\right)$
(f) (3h, 0), (2h, k) and (h, 2k)

3. Find the area of the quadrilateral whose vertices are given below.

(a) (0, 0), (4, 0), (4, 6), (0, 6)
(b) (b) (6, 2), (5, 3), (3, 0) and (1, 2)
(c) P(3, 4), Q(4, –7), R(1, 1) and S(5, –2)
(d) (d) A(–5, 1), B(5, 5) and C(10, 7)

(e) P (6, 8), Q(0, 10), R(4, –2) and S(6, –4)

4. (a) If P (3, 4), Q (1, 2), R (7, 2) are three points and S (x, y) is another point then prove that $\frac{\Delta PQR}{\Delta PQR} = \frac{2}{y-2}$

(c) If the point (m, 1), (1, 2) and (0, n+1) are collinear than prove that $\frac{1}{m} + \frac{1}{n} = 1.$

(d) If (a, 0), (0, b) and (3, 3) are collinear points than prove that $\frac{1}{a} + \frac{1}{b} = \frac{1}{3}.$

(e) If $\left(\frac{-a}{b}, 0\right)$, (x, y) and (0, a) are collinear then prove that y = bx+a.

(f) If A, B, C are the middle points of QR, RP and PQ of a triangle PQR with P(1, –4), Q (5, 6) and R(–3, 2) respectively then prove that

(i) Ar (ΔAQR) = 4 Ar(ΔABC) (ii) Ar (ΔABC) = Ar(ΔPAC)

5. (a) The coordinates of a quadrilateral are A (6, 3), B (–3, 5), C(4, –2) and D(k, 3K). If Ar (ΔABC) = 2Ar (ΔDBC), find value of k.

(b) The coordinates of vertices of a quadrilateral are P (6, k), Q (–3, 5), R (4, –2) and S (k, 2k). If $\frac{\Delta QRS}{\Delta PQR} = \frac{1}{2}$ find value of k.

(c) A (a, a+1), B (0, 7), C (2, –1) and D (3, –2) are three vertices of a quadrilateral PQRS. If Ar (ABCD) = 8. Ar (ΔACD), find value of a.

(d) if A (6, 3), B (–3, 5), C(4, –2), D(x, y) be coordinates of four vertices of a quadrilateral then prove that $\frac{\Delta DBC}{\Delta ABC} = \frac{x+y-2}{7}.$

6. (a) If A(1, 2), B(4, 2), C(5, 4) and D(x, y) be the four vertices of a parallelogram ABCD. Find value of (x, y) and area of parallelogram.

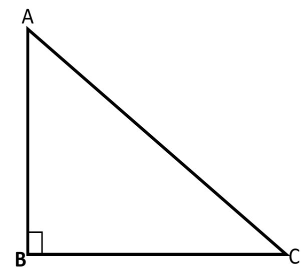
(b) Let A (5, –1), B(1, –3) and D(1, –5) are coordinates of vertices of a triangle If L, M, N be middle point of BC, CA and AB.

(i) Find area of ΔABC and ΔPQR. (ii) Show that area of ΔPQR = Ar (ΔAQR.)

Unit 5

Trigonometry

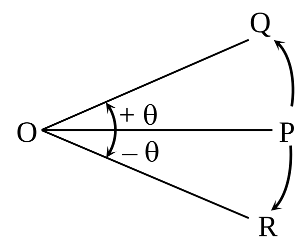
5.0 Review



Divide all the students in groups of three and distribute one right angled triangle ABC to each group. Ask them to measure angles and sides of that triangle. Identify p, b, h and d find the ratio of the two different sides according to the reference angle A.

Identify p, b, h according to the angle C, and find the ratios. Prepare a group report and present in classroom.

5.1 Measurement of Angles

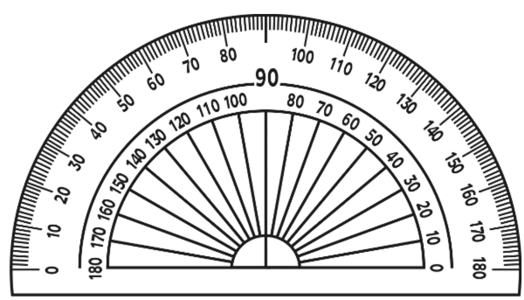


In the figure, O is a fixed point and OP is an initial line. OP revolves in anti-clockwise direction and the angle is formed in the position of Q Then the angle of rotation of OP about O at a point Q is denoted by ∠POQ and ∠POQ = θ, where θ is the Greek alphabet that denotes the arbitrary measures.

The amount of rotation of a revolving line about a fixed point with respect to the initial line is called an angle**.**

If the revolving line rotates about a fixed point from its initial line in anti-clockwise direction, the angle made by the lines is called a positive angle (+ θ), otherwise it is called negative angle (– θ).

Measuring Systems of Angles



How many systems are there for measuring an angle nowadays?

An angle is generally measured in degrees. In trigonometry, the systems of measurement of an angle are as follow;

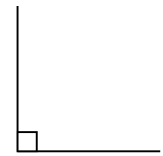
(i) Sexagesimal system (Degree measure)

(ii) Centesimal system (Grade measure)

(iii) Circular system (Radian measure)

(i) Sexagesimal system (Degree measure)

The sexagesimal system is more familiar unit of measurement of an angle. It measures in degree (°) unit. This system was initially used in the British. So, it is also called the British System. In this system, a right angle is divided into 90 equal parts, each part is called a degree.



A Right Angle

Therefore, ***1 right angle = 90°.***

Furthermore, a degree is again divided into 60 equal parts, each part is called a minute (′).

Therefore, ***1° = 60′.***

Similarly, a minute is again divided into 60 equal parts, each part is called a second (″).

Therefore, ***1′ = 60″.***

Hence, 1 *right angle* = 90° = (90 × 60)′ = 5400′ = (5400 × 60)″ = 324000″ .

Example 1

How many sexagesimal minutes are there in 23°?

Solution:

Here, 23° = (23 x 60)' [∴ 1° = 60°]

= 1380'

∴ There are 1380 sexagesimal minutes in 23°.

Example 2

Convert into sexagesimal seconds: 25°19′ 30″

Solution

Here, 25°19′ 30"

= (25 × 60 × 60 + 19 × 60 + 30)″ [∵ 1° = 60′ = 60 × 60″ ; 1′ = 60″]

= (90000 + 1140 + 30)″ = 91170″

Example 3

Convert into degree: 25°19' 30"

Solution: Here, 25°19' 30"

$= \left(25 + \frac{19}{60} + \frac{30}{60\times60}\right)°$ $\left[\because 1' = \left(\frac{1}{60}\right)° \text{ and } 1'' = \left(\frac{1}{60\times60}\right)°\right]$

$= (25 + \frac{19}{60} + \frac{1}{120})°$

$= (\frac{3000+38+1}{120})° = (\frac{3039}{120})° = 25.325°$

(ii) Centesimal system (Grade measure)

The centesimal system measures the angle in grade (^{g}). It is specially used in France. So, it is also called French system. In this system, a right angle is divided into 100 equal parts, each part is called grade (^{g}).

Therefore, ***1 right angle = 100^g.***

A grade is again divided into 100 equal parts, each part is called a minute (').

Therefore, ***1^g = 100 '.***

Similarly, a minute is divided into 100 equal parts, each part is called second (").

Therefore, ***1' = 100".***

Hence, 1 right angle = 100^g = (100 × 100)' = 10000' = (10000 × 100) " = 1000000".

Example 4

How many centesimal minutes are there in 35^g?

Solution

Here, 35^g = (35 x 100)'

= 3500'

∴ There are 3500' centesimal minutes in 35^g.

Example 5

Convert into centesimal seconds: 65^{g}36' 97"

Solution:

Here, 65^{g}36' 97"

= (65 × 100 × 100 + 36 × 100 + 97)"

[∵ 1' = 100"; 1° = 100' = (100 × 100)" = 10000"]

= (650000 + 3600 + 97)" = 653697"

Example 6

Convert into grade: $25^g 19'\ 30''$

Solution:

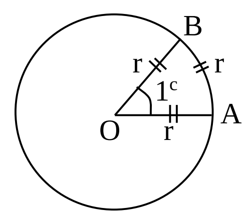
Here, $25^g\ 19'\ \ 30''$

$(25 + \frac{19}{100} + \frac{30}{100\times100})^g$

$[\because 1' = \left(\frac{1}{100}\right)^g$ and $1'' = \left(\frac{1}{100\times100}\right)^g]$

$= (25 + \frac{19}{100} + \frac{30}{10000})^g = (\frac{250000+1900+30}{10000})^g = (\frac{251930}{10000})^g = 25.193^g$

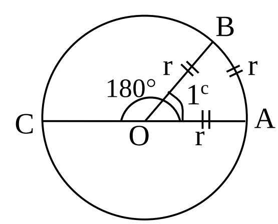
(iii) Circular System (Radian measure)



In this System, an angle is measured in radian (^{c}). A radian is considered as the unit for the measurement of central angle inscribed by an arc of a circle equal in length to its radius. In the figure, O is the centre of a circle and $OA = OB = \widehat{AB} = r$ units. Then, $\angle AOB = 1$ radian (1^c). When the radius OA rounds in one complete rotation, it makes $2\pi^c$, where π is the constant quantity, so a right angle is $\frac{2\pi^c}{4} = \frac{\pi^c}{2}$.

An angle subtended at the center of a circle by an arc whose length is equal to its radius is called a radian.

Theorem 1: A radian is a constant angle.



Proof: In the figure, O is the centre of the circle and $\angle AOB$ is the angle subtended at O by the arc equal to the radius (r).

i.e., $OA = OB = \widehat{AB} = r$ units.

Now, according to the definition of the radian, $\angle AOB = 1^c$.

If AO is produced to a point C on the circumference, then $\angle AOC = 180°$ and the semi-circle ABC = πr units.

We know that,

$\frac{\angle AOB}{\angle AOC} = \frac{\widehat{AB}}{\widehat{ABC}}$

[∵ The central angles and their corresponding arcs in proportion.]

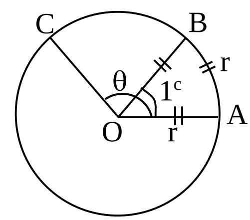
or, $\frac{1^c}{180°} = \frac{r}{\pi r}$

or, $1^c = \frac{180°}{\pi}$

Here, 180° and π are both constants. Hence, a radian is a constant angle.

Theorem 2

If θ is the central angle, l is the length of corresponding arc of the central angle θ and r is the radius of a circle then, $\theta = \left(\frac{l}{r}\right)^c$.



Proof: Let O be the centre of a circle in which OA = r units be its radius and take a point B on its circumference such that OA = AB (r). Then by the definition of the 1 radian, we have ∠AOB = 1^c.

Also, take another point C, then the arc ABC substances the angle of θ at its center i.e. ∠AOC = θ^c and $\widehat{ABC}$= l units.

Since the angles at the centre of a circle are proportional to their corresponding arcs on which they stand so,

$\frac{\angle AOC}{\angle AOB} = \frac{\widehat{ABC}}{\widehat{AB}}$ [∵ The central angles and their corresponding arcs are proportion.]

or, $\frac{\theta}{1^c} = \frac{l}{r}$

$\therefore \theta = \left(\frac{l}{r}\right)^c$

Relation between Degree, Grade and Radian

We have, a right angle has 90° or 100^g or $\frac{\pi^c}{2}$. Then, the relation among degree, grade and radian is as follows:

$90° = 100^g$	$90° = \frac{\pi^c}{2}$	$100^g = 90°$	$100^g = \frac{\pi^c}{2}$	$\frac{\pi^c}{2} = 90°$	$\frac{\pi^c}{2} = 100^g$
$1° = \left(\frac{100}{90}\right)^g$ $\therefore 1° = \left(\frac{10}{9}\right)^g$	$1° = \left(\frac{\pi}{180}\right)^c$	$1^g = \left(\frac{90}{100}\right)^o$ $\therefore 1^g = \left(\frac{9}{10}\right)^o$	$1^g = \left(\frac{\pi}{200}\right)^c$	$1^c = \left(\frac{180}{\pi}\right)^o$	$1^c = \left(\frac{200}{\pi}\right)^g$

Example 7

Change the angle 60^g 50' 10'' into radian.

Solution: Here,

60^g 50' 10'' = $\left(60 + \frac{50}{100} + \frac{10}{10000}\right)^g$

$= (60 + 0.5 + 0.001)^g$

$= 60.501^g$

$= 60.501 \times \left(\frac{\pi}{200}\right)^c$

$= \frac{60501\ \pi^c}{200000}$

$= \frac{60501\ \pi^c}{200000}$

Example 8

Convert $\frac{5\pi^C}{32}$ into sexagesimal system:

Solution

Here, $\frac{5\pi^c}{16} = \frac{5\pi}{32} \times \frac{180°}{\pi}$

$= 28.125°$

$= 28° (0.125 \times 60)'$

$= 28°\ 7.5'$

$= 28°\ 7' (0.5 \times 60)''$

$= 28°\ 7'\ 30''$

Example 9

Change the angle $50°30'$ into radian.

Solution

Here, $50°30' = \left(50 + \frac{30}{60}\right)^o$

$= \left(50 + \frac{1}{2}\right)^o$

$= \left(\frac{100+1}{2}\right)^o$

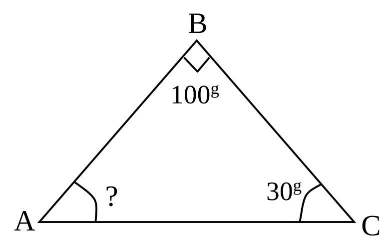
$= \left(\frac{101}{2}\right)^o$

$= \left(\frac{101}{2} \times \frac{\pi}{180}\right)^c \qquad = \left(\frac{101\pi}{360}\right)^c$

Example 10

One angle of a right-angled triangle is 27°. Find its third angle in grade measure.

Solution



In a right-angled triangle ABC, $\angle B = 100^g$,

$\angle C = 27° = 27 \times \left(\frac{10}{9}\right)^g = 30^g$, $\angle A = ?$

Now, we know that

$\angle A + \angle B + \angle C = 200^g$ [∴ *Sum of interior angles of triangle in grade*]

or, $\angle A + 100^g + 30^g = 200^g$

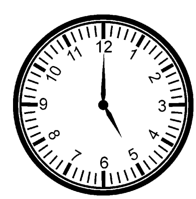
or, $\angle A = 200^g - 130^g = 70^g$

∴ The required third angle of the triangle is 70^g.

Example 11

Find the angle made by two hands of a clock at 5 O'clock in grades.

Solution



Here, at 5 O'clock, the difference between the two hands of the clock is 25 minutes.

Now, we have 60 minutes = 360° [∵ *One complete rotation*]

or, 1 minute $= \frac{360^o}{60} = 6°$

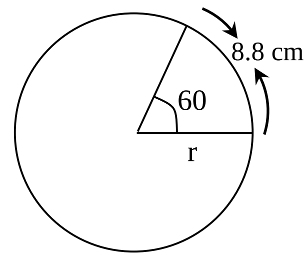
∴ 25 minutes = 25 × 6° = 150°

Hence, the measure of required angle in grade is $150 \times \left(\frac{10}{9}\right)^g = \left(\frac{1500}{9}\right)^g = \left(166\ \frac{2}{3}\right)^g$.

Example 12

If the arc of 8.8 cm subtends an angle of 60° at the centre of a circle. Find its radius. ($\pi = \frac{22}{7}$)

Solution:



Here, Length of arc (l) = 8.8 cm,

Measure of central angle (θ) = 60°

$$= 60 \times \frac{\pi^c}{180} = \frac{\pi^c}{3}$$

Radius of circle (r) = ?

Now, we know that

$$\theta = \frac{l}{r}$$

$$\text{or, } \frac{\pi}{3} = \frac{8.8}{r}$$

$$\text{or, } r = 8.8 \times \frac{3}{\pi} = 8.8 \times \frac{3}{\frac{22}{7}} = 8.8 \times 3 \times \frac{7}{22} = 8.4 \text{ cm}$$

Hence, the length of its radius is 8.4 cm.

Example 13

If D is the number of degrees and G is the number of grades of the same angle, prove that: $\frac{D}{9} = \frac{G}{10}$

Solution: We have,

1 right angle = 90° = 100^g

Let the angle be x right angle. Then, D = 90x and G = 100x.

Now, D = 90x

or, $x = \frac{D}{90}$ ………… (i) and

G = 100x

or, $x = \frac{G}{100}$ …… (ii)

From (i) and (ii), we get

$$\frac{D}{90} = \frac{G}{100}$$

or, $\frac{D}{9} = \frac{G}{10}$.

Exercise 5.1

1. **Answer the following questions in single sentence:**
   (a) List the measuring systems of an angle.
   (b) How many sexagesimal seconds are there in 30°30' ?
   (c) How many grades are there in 400' ?
   (d) Express the number of grades in 81°.
   (e) Write the measure of a right angle in radian.
   (f) How many radians are there in $\frac{3}{4}$ of a right angle?
   (g) What is the value of 40% of a right angle in radian?
2. **Express the following angles into sexagesimal seconds:**
   (a) 55°30'' (b) 10° 15' 25'' (c) 55° 56' 28''
3. **Convert into degree:**
   (a) 36°30' (b) 25° 15' 30'' (c) 48° 50' 45''
4. **Change into centesimal seconds:**
   (a) 25^g 29'' (b) 25^g 34' 29'' (c) 25^g 74' 99''
5. **Express into grade:**
   (a) 36^g 30' (b) 27^g 28' 30'' (c) 79^g 47' 23''
6. **Convert the following angles into degree:**
   (a) 42^g 50' (b) 35^g 65' 45'' (c) 85^g 44' 50''
7. **Change the following angles into degree:**
   (a) 75° 30' (b) 42° 45' 15'' (c) 85° 24' 40''
8. **Change the following angles into radian:**
   (a) 60° 45' (b) 57^g 49' 87'' (c) 66° 36' 35''
9. **Express the following angles into sexagesimal system:**
   (a) $\frac{\pi}{8}^c$ (b) $\frac{2\pi}{7}^c$ (c) $\frac{3\pi}{32}^c$

10. Change the following angles into centesimal system:

(a) ${\frac{\pi}{16}}^c$ (b) ${\frac{4\pi}{25}}^c$ (c) ${\frac{5\pi}{21}}^c$

11. (a) Find the measure of the third angle in degree of a triangle having two angles 30^g and 81^g.

(b) Find the measure of the third angle in degree of a triangle having two angles $40°$ and 63^g.

(c) Find the measure of the third angle in grade of a triangle having two angles $54°$ and 36^g.

(d) Find the measure of the forth angle in grade of a quadrilateral having three angles ${\frac{\pi}{5}}^c$, 36^g and 45^g.

(e) Find the measure of the third angle in radian of a right triangle with an angle $45°$.

12. (a) One angle of a triangle is 30^g. If the ratio of the remaining two angles is 3:7, find all angles of the triangle in degree.

(b) The angles of triangle are in the ratio 4:5:9. Find the angles in radian.

(c) If the angles of a quadrilateral are in the ratio 1:2:3:4, find all the angles in grades.

(d) Divide $63°$ into two parts such that the ratio of their grades measure is 2:5.

(e) Find the ratio of $60°$ and 72^g.

13. Find the angle in degree, grade and radian formed by the minute hand and hour hand of a clock at:

(a) Half past 3 (b) Quarter past 6 (c) Quarter to 2

14. (a) Find the central angle in centesimal measure subtended at the center of a circle of radius 6 cm by an arc of 24 cm long.

(b) The radius of a circle is 21 cm. Find the length of arc of the circle which subtends an angle of $45°$ at its center.

(c) The arc of the length 28 cm subtends an angle of 72^g at the center of a circle. Find the length of the diameter of the circle.

(d) A man running along a circular track at the rate of 20 km per hour travels on the track in 15 seconds which subtends $60°$at the center. Find the diameter of the circle.

(e) A pendulum 50 cm long vibrates 2° 30′ each side of its standard position. Find the length of the arc through which it swings.

(f) The minute hand of a clock is 7 cm. How far does the tip of the hand move in 20 minutes?

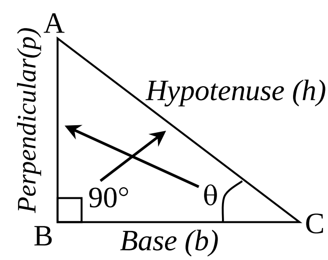
15. (a) If D is the number of degrees and G is the number of grades, prove that $\frac{D}{G} = \frac{9}{10}$.

(b) If G, D and R denote the number of grades, degrees and radian respectively of an angle, prove that: $\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$.

(c) If α and β denote the number of sexagesimal and centesimal second of any angle respectively, prove that: α:β = 81:250.

5.2 Trigonometric Ratios

What is ratio? How many ratios can be formed from the sides of the right-angled triangle?



In the given right-angled triangle ABC, ∠ABC = 90°

∴ Its opposite side (AC) = Hypotenuse (h).

Reference angle ∠ACB = θ

∴ Its opposite side (AB) = Perpendicular (p)

Remaining side (BC) = Base (b)

Now, the sides of right angled ΔABC are $\frac{AB}{AC}$, $\frac{BC}{AC}$ and $\frac{AB}{BC}$ i.e., $\frac{p}{h}$, $\frac{b}{h}$ and $\frac{p}{b}$, are called **fundamental trigonometric ratios.**

The ratios $\frac{AC}{AB}$, $\frac{AC}{BC}$ and $\frac{BC}{AB}$ i.e., $\frac{h}{p}$, $\frac{h}{b}$ and $\frac{b}{p}$, are called **reciprocal trigonometric ratios of** fundamental ratios**.** They are defined as follow:

(i) The ratio of the perpendicular to the hypotenuse of a right-angled triangle is called sine of the reference angle θ. It is symbolized as $\mathbf{\sin\theta = \frac{p}{h}}$.

(ii) The ratio of the base to the hypotenuse of a right-angled triangle is called cosine of the reference angle θ. It is denoted by $\mathbf{\cos\theta = \frac{b}{h}}$.

(iii) The ratio of the perpendicular to the base of a right-angled triangle is called tangent of the reference angle θ. It is denoted by $\mathbf{\tan\theta = \frac{p}{b}}$.

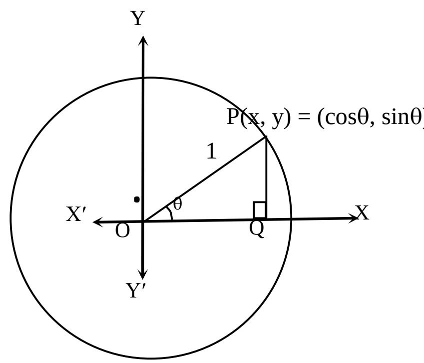
(iv) The ratio of the base to the perpendicular of a right-angled triangle is called cotangent of the reference angle θ. It is denoted by **cot θ =** $\mathbf{\frac{b}{p}}$.

(v) The ratio of the base to the hypotenuse of a right-angled triangle is called secant of the reference angle θ. It is symbolized as **sec θ =** $\mathbf{\frac{h}{b}}$.

(vi) The ratio of the hypotenuse to the perpendicular of a right-angled triangle is called cosecant of the reference angle θ. It is denoted by **cosec θ =** $\mathbf{\frac{h}{p}}$.

Alternatively

In the adjoining figure, ΔOQP is a right angled triangle in which hypotenuse is the radius of unit circle with centre O(0, 0). P(x, y) be a point on a circumference of circle. Then OP = 1, ∠POQ = θ, OQ = x and PQ = y. Then,



$$\cos\theta = \frac{b}{h} = \frac{OQ}{OP}$$

$$= \frac{OQ}{1}$$

∴ OQ = cos θ, i.e., x = cos θ

$$\sin\theta = \frac{p}{h} = \frac{PQ}{OP}$$

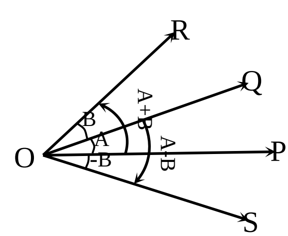
$$= \frac{PQ}{1}$$

∴ PQ = sin θ, i.e., y = sin θ

Hence, the coordinates of P will be (x, y) = (cosθ, sinθ)

Example1

If 5cosθ = 4, find the trigonometric ratios sin θ and tanθ.



Solution: Here,

5cosθ = 4

or, $\cos\theta = \frac{4}{5} = \frac{b}{h}$

Now, $h^2 = p^2 + b^2$

or, $p^2 = h^2 - b^2$

or, $p = \sqrt{h^2 - b^2} = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3$

Now, we know that

$\sin\theta = \frac{p}{h} = \frac{3}{5}$ and $\tan\theta = \frac{p}{b} = \frac{3}{4}$.

Example 2

If $\sqrt{3}\tan\theta = 1$, find the value of $\frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}$.

Solution:

Here, $\sqrt{3}\tan\theta = 1$

$\tan\theta = \frac{1}{\sqrt{3}} = \frac{p}{b}$

Now, we know that

$h^2 = p^2 + b^2$

$\therefore\ h = \sqrt{p^2 + b^2} = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{1+3} = \sqrt{4} = 2$

$\therefore\ \sin\theta = \frac{p}{h} = \frac{1}{2}$ and $\cos\theta = \frac{b}{h} = \frac{\sqrt{3}}{2}$

Now, we have

$$\frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \frac{\frac{1}{2} - \frac{\sqrt{3}}{2}}{\frac{1}{2} + \frac{\sqrt{3}}{2}} = \frac{\frac{1-\sqrt{3}}{2}}{\frac{1+\sqrt{3}}{2}} = \frac{1-\sqrt{3}}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$= \frac{(1-\sqrt{3})^2}{(1)^2 - (\sqrt{3})^2} = \frac{1-2\sqrt{3}+3}{1-3} = \frac{4-2\sqrt{3}}{-2} = \frac{2(2-\sqrt{3})}{-2} = \sqrt{3} - 2$$

Example 3

If $\sin A = \frac{4}{5}$ and $\sin B = \frac{12}{13}$, find the values of $\sin A \times \cos B - \cos A \times \sin B$.

Solution

Here, $\sin A = \frac{4}{5} = \frac{p}{h}$

Now, we know that

$h^2 = p^2 + b^2$

or, $b^2 = h^2 - p^2$

or, $b = \sqrt{h^2 - p^2} = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3$

$\therefore$ cosA = $\frac{b}{h} = \frac{3}{5}$

And sinB = $\frac{12}{13} = \frac{p}{h}$

$\therefore$ b = $\sqrt{h^2 - p^2}$ = $\sqrt{13^2 - 12^2}$ = $\sqrt{169 - 144}$ = $\sqrt{25}$ = 5

$\therefore$ cosB = $\frac{b}{h} = \frac{5}{13}$

Now, we have

sinA x cosB – cosA x sinB $= \frac{4}{5} \times \frac{5}{13} - \frac{3}{5} \times \frac{12}{13}$

$= \frac{20}{65} - \frac{36}{65}$

$= \frac{20-36}{65} = -\frac{16}{65}$

Exercise 5.2

1. Answer the following in single sentence:

(a) What is the ratio of cos θ ?

(b) Write the fundamental trigonometric ratios.

(c) Write the product of sin θ and cosec θ.

(d) Write cos θ in terms of sec θ.

(e) Write tan θ in terms of sin θ and cos θ.

2. (a) If sin A = $\frac{3}{5}$, find the values of the remaining trigonometric ratios.

(b) If cos A = $\frac{12}{13}$, find the values of sin A, tan A and cosec A.

(c) If 17 cos θ = 8, find the ratios of sin θ, cot θ and cosec θ.

(d) If tan α = $\frac{2\sqrt{a}}{a-1}$, find sin α and cos α.

(e) If cosec x = $\sqrt{2}$, find the value of cos x and tan x.

3. (a) If cosec y = $\frac{13}{12}$, find the value of 3 cot y – 2 tan y.

(b) If tan θ = $\frac{2}{3}$, find the value of $\frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$.

(c) If cot θ = $\frac{4}{3}$, find the value of $\frac{3\sin\theta - 2\cos\theta}{2\sin\theta + 3\cos\theta}$.

4. If $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, find the values of the following trigonometric expressions.

(a) $\sin A \times \cos B + \cos A \times \sin B$ (b) $\cos A \times \cos B - \sin A \times \sin B$

(c) $\sin A \times \cos B - \cos A \times \sin B$ (d) $\cos A \times \cos B + \sin A \times \sin B$

(e) $\frac{\tan A + \tan B}{1 - \tan A . \tan B}$ (f) $\frac{\tan A - \tan B}{1 + \tan A . \tan B}$

5. (a) If $\sec A = \frac{m^2 + n^2}{m^2 - n^2}$, find he values of sin A and cot A.

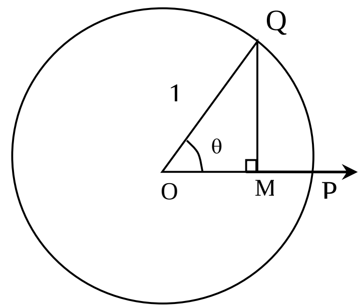
(b) If $\cot x = \frac{p}{q}$, prove that $\frac{p\cos x - q\sin x}{p\cos x + q\sin x} = \frac{p^2 - q^2}{p^2 + q^2}$.

(c) If $\cos \theta = \frac{x}{\sqrt{x^2 - y^2}}$, prove that $x \sin A + y \cos \theta = \sqrt{x^2 - y^2}$.

(d) If sin x = m and tan x = n, prove that, prove that $\frac{1}{m^2} - \frac{1}{n^2} = 1$.

5.3 Trigonometric Ratios of Standard Angles

(i) Values of Trigonometric Ratios of 0° and 90°



In the adjoining figure, the revolving line OP of the length 1 unit makes an angle θ at the position of OQ. Draw QM⊥OP. Then QOM is a right-angled triangle in which ∠QMO = 90° and ∠QOM = θ. When the angle θ becomes smaller and smaller and the line segment QM also becomes smaller in length. In figure, when θ becomes 0°, the point Q coincides with M, then QM = 0 and OQ = OM.

Now, we have

(i) $\sin 0° = \frac{QM}{OQ} = \frac{0}{OQ} = 0$ $[\because \sin\theta = \frac{p}{h}]$

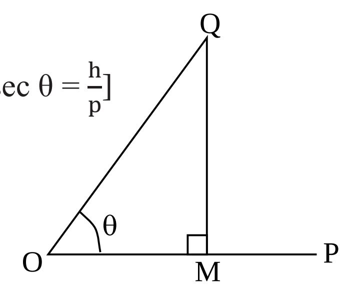
(ii) $\cos 0° = \frac{OM}{OQ} = \frac{OQ}{OQ} = 1$ $[\because \cos\theta = \frac{b}{h}]$

(iii) $\tan 0° = \frac{QM}{OM} = \frac{0}{OM} = 0$ $[\because \tan\theta = \frac{p}{b}]$

(iv) $\cot 0° = \frac{OM}{QM} = \frac{OM}{0} = \infty$ (Undefined) $[\because \cot\theta = \frac{b}{p}]$

i(v) $\sec 0° = \frac{OQ}{OM} = \frac{OM}{OM} = 1$ $[\sec\theta = \frac{h}{b}]$

(vi) $\text{cosec } 0° = \frac{OQ}{QM} = \frac{OQ}{0} = \infty$ (Undefined) $[\because \text{cosec }\theta = \frac{h}{p}]$



Again, if the value of θ is continuously increasing approaching to 90°, the point O coincides with the point M. i.e. OM = 0 and OR = QM.

Now, we have,

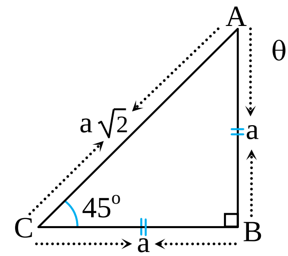
(i) $\sin 90° = \frac{QM}{OQ} = \frac{OQ}{OQ} = 1$ (ii) $\cos 90° = \frac{OM}{OQ} = \frac{0}{OQ} = 0$

(iii) $\tan 90° = \frac{QM}{OM} = \frac{OQ}{0} = \infty$ (iv) $\cot 90° = \frac{OM}{QM} = \frac{0}{OQ} = 0$

(v) $\sec 90° = \frac{OQ}{OM} = \frac{OM}{0} = \infty$ (vi) $\text{cosec } 90° = \frac{OQ}{QM} = \frac{OQ}{OQ} = 1$

(ii) Values of Trigonometric Ratios of 45°

Let, ABC is an isosceles right-angled triangle in which AB = BC = a (suppose), $\angle ABC = 90^o$ and $\angle ACB = \angle BAC = 45^o$.



$AC = \sqrt{AB^2 + BC^2}$ [∵ *Pythagoras theorem,* $h^2 = p^2 + b^2$]

$= \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}$.

Now, we have

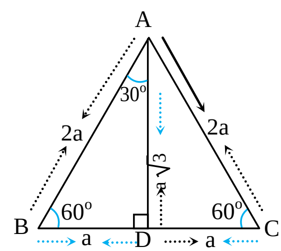
(i) $\sin 45^o = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$ (ii) $\cos 45^o = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$

(iii) $\tan 45^o = \frac{AB}{BC} = \frac{a}{a} = 1$ (iv) $\cot 45^o = \frac{BC}{AB} = \frac{a}{a} = 1$

(v) $\sec 45^o = \frac{AC}{BC} = \frac{a\sqrt{2}}{a} = \sqrt{2}$ (vi) $\text{cosec } 45^o = \frac{AC}{AB} = \frac{a\sqrt{2}}{a} = \sqrt{2}$

(iii) Values of Trigonometric Ratios of 30° and 60°

Let, ABC is an equilateral triangle in which, AB = BC = CA = 2a and $\angle CAB = \angle ABC = \angle BCA = 60°$. Draw $AD \perp BC$, then $\angle BAD = 30°$ and BD = DC = a. From the right-angled triangle ABD, we have



$AD = \sqrt{AB^2 - BD^2}$ [∵ *Pythagoras theorem,* $h^2 = p^2 + b^2$]

$= \sqrt{4a^2 - a^2} = \sqrt{3a^2} = a\sqrt{3}$.

Now, we have

For ratios of 30°:

(i) $\sin 30^o = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$ (ii) $\cos 30^o = \frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$

(iii) $\tan 30^o = \frac{BD}{AD} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$ (iv) $\cot 30^o = \frac{AD}{BD} = \frac{a\sqrt{3}}{a} = \sqrt{3}$

(v) $\sec 30^o = \frac{AB}{AD} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}$ (vi) $\text{cosec} 30^o = \frac{AB}{BD} = \frac{2a}{a} = 2$

For ratios of 60º:

(i) $\sin 60^o = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$

(ii) $\cos 60^o = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$

(iii) $\tan 60^o = \frac{AD}{BD} = \frac{a\sqrt{3}}{a} = \sqrt{3}$

(iv) $\text{cosec } 60^o = \frac{AB}{AD} = \frac{2a}{\sqrt{3}a} = \frac{2}{\sqrt{3}}$

(v) $\sec 60^o = \frac{AB}{BD} = \frac{2a}{a} = 2$

(vi) $\cot 60^o = \frac{BD}{AD} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$

Value of trigonometric ratios of some standard angles (0º, 30º, 45º, 60º, 90º)

Angle / Ratio	0º	30º	45º	60º	90º
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Example 1

Find the value of (sin 60° + cos 30°) tan 30°.

Solution: Here,

$$(\sin 60° + \cos 30°) \tan 30° = \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)\frac{1}{\sqrt{3}} = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}} = 1$$

Example 2

Find the value of $\sin^2 30° - \cos^2 60° + \tan^3 45°$.

Solution

Here, $\sin^2 30° - \cos^2 60° + \tan^3 45° = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + (1)^3 = \frac{1}{4} - \frac{1}{4} + 1 = 1$

Example 3

Prove that: $\tan^2 30° + 2 \sin 60° + \tan^2 45° - \tan 60° + \cos^2 30° = 2\frac{1}{12}$

Solution

Here, LHS = $\tan^2 30° + 2 \sin 60° + \tan^2 45° - \tan 60° + \cos^2 30°$

$$= \left(\frac{1}{\sqrt{3}}\right)^2 + 2 \times \frac{\sqrt{3}}{2} + (1)^2 - \sqrt{3} + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{3} + \sqrt{3} + 1 - \sqrt{3} + \frac{3}{4}$$

$$= \frac{1}{3} + 1 + \frac{3}{4}$$

$$= \frac{4 + 12 + 9}{12} = \frac{25}{12}$$

$$= 2\frac{1}{12} = \text{RHS.}$$

Example 4

Find the value of $\dfrac{\tan^2 \frac{\pi^c}{3} \times \operatorname{cosec} \frac{\pi^c}{6} \times \tan \frac{\pi^c}{4}}{\text{Sec}^2 \frac{\pi^c}{4} \times \sec \frac{\pi^c}{3} \times \sin \frac{\pi^c}{6} \times \sin^2 \frac{\pi^c}{4}}$

Solution:

Here, $\dfrac{\tan^2 \frac{\pi^c}{3} \times \operatorname{cosec} \frac{\pi^c}{6} \times \tan \frac{\pi^c}{4}}{\text{Sec}^2 \frac{\pi^c}{4} \times \sec \frac{\pi^c}{3} \times \sin \frac{\pi^c}{6} \times \sin^2 \frac{\pi^c}{4}}$

$$= \frac{\tan^2 \frac{180°}{3} . \operatorname{cosec} \frac{180°}{6} . \tan \frac{180°}{4}}{\sec^2 \frac{180°}{4} . \sec \frac{180°}{3} . \sin \frac{180°}{6} \sin^2 \frac{180°}{4}}$$

$$= \frac{\tan^2 60° . \operatorname{cosec} 30° . \tan 45°}{\sec^2 45° . \sec 60° . \sin 30° . \sin^2 45°}$$

$$= \frac{(\sqrt{3})^2 \times 2 \times 1}{(\sqrt{2})^2 \times 2 \times \frac{1}{2} \times \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{3 \times 2}{2 \times \frac{1}{2}} = 6$$

Example 5

Find the value of $1 - 2 \sin^2 30° = \left(\frac{1 - \tan^2 30°}{1 + \tan^2 30°}\right)$.

Solution: Here,

$$\text{LHS} = 1 - 2 \sin^2 30° = 1 - 2\left(\frac{1}{2}\right)^2 = 1 - 2 \times \frac{1}{4} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\text{RHS} = \left(\frac{1 - \tan^2 30°}{1 + \tan^2 30°}\right) = \left(\frac{1 - \left(\frac{1}{\sqrt{3}}\right)^2}{1 + \left(\frac{1}{\sqrt{3}}\right)^2}\right) = \left(\frac{1 - \frac{1}{3}}{1 + \frac{1}{3}}\right) = \left(\frac{2}{3} \times \frac{3}{4}\right) = \frac{1}{2}$$

Hence, LHS = RHS.

Exercise 5.3

1. Write the answer in one sentence:

- **(a)** What is the value of sin 45°?
- **(b)** Which trigonometric ratios of standard angles have the value $\frac{1}{2}$?
- **(c)** Write the value of tan 30°?
- **(d)** What is the value of sin 90° + cos 0° ?
- **(e)** What is the value of tan 45° – cos 0° ?

2. **Evaluate:**

(a) $\sin 45°.\cos 45° - \cos^2 60°$

(b) $\cos 30°.\cos 45° - \sin 30°.\sin 45°$

(c) $2\sin 60°.\sin 90° + \cos 60°.\cos 0°$

(d) $\dfrac{\sin 60° + \cos 30°}{\sin 90° + \sin 30° + \cos 60°}$

(e) $2\sin^2 30° - 3\cos^2 45° + \tan^2 60°$

(f) $3\tan^2 45° - \sin^2 60 - \cot^2 30 + \sec^2 45°$

(g) $4\sin^2 60° + 3\tan^2 30° - 8\sin 45°.\cos 45°$

(h) $\cot^2 45° + \text{cosec}^2 45°$

(i) $\dfrac{\cos^2 60° - \sin^2 60°}{\cos 30° + \cos 45°}$

(j) $\dfrac{2}{3}\sin^2 60^o + 3\tan^2 30^o + \dfrac{4}{3}\sin^2 45°$

3. **If θ = 30°, prove that:**

(a) $\cos 2\theta = \sin \theta$ (b) $\cos 2\theta = 1 - 2\sin^2\theta$

(c) $\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$ (d) $\sin 2\theta = 2\sin\theta.\cos\theta$

4. **Prove that:**

(a) $\sin(45° - 30°) = \sin 45°.\cos 30° - \cos 45°.\sin 30°$

(c) $\dfrac{\tan \frac{\pi^c}{4} - \tan \frac{\pi^c}{6}}{1 + \tan \frac{\pi^c}{4} . \tan \frac{\pi^c}{6}} = 2 - \sqrt{3}$ (b) $\cot(45° + 30°) = \dfrac{\cot 30^o.\cot 45^o - 1}{\cot 30^o + \cot 45^o}$

(d) $\dfrac{1 - \tan^2 \frac{\pi^c}{6}}{1 + \tan^2 \frac{\pi^c}{6}} = \cos\dfrac{\pi^c}{3}$

5. **If $\alpha = 30^o$, $\beta = 45^o$, $\theta = 60^o$ then verify that:**

(a) $\sin(\alpha + \theta) = \sin \alpha.\cos \theta + \cos \alpha.\sin \theta$ (b) $\tan 2\beta = \dfrac{2\tan\beta}{1 - \tan^2\beta}$

(c) $4 \sin^2\theta + 3 \tan^2\alpha - 8 \sin\beta.\cos\beta = 0$ (d) $\sin 3\theta = 3\sin \theta - 4\sin^3 \theta$

(e) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ (f) $\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$

6. Find the value of x:

(a) $\tan^2 45° \,(-\text{cosec } 60°) = x \cos 45°.\sin 45°.\cot 60°$

(b) $3\sin 60° + x.\cos 30° \tan 45° = x \cot 30°$

(c) $12x \tan^2 45° - 12 \sin^2 60° - 6\cot^2 30° + 4\sec^2 45° = 17$

(d) $\sin 30° + 2\cot^2 30° + x \cos^2 30° = 8 + \tan^2 45° + \cos 60°$

(e) $x + 3\tan^2 30° + 4\cos^2 30° = 2\sec^2 45° + 4 \sin^2 60°$

5.4 Identities of Trigonometric Ratios

Relation of Trigonometric Ratios

A. Reciprocal Relations

(i) cosec θ is the reciprocal ratio of sin θ.

$\sin\theta \times \text{cosec }\theta = \frac{p}{h} \times \frac{h}{p} = 1.$

$\therefore$ $\sin\theta \times \text{cosec }\theta = 1 \Rightarrow \sin\theta = \frac{1}{\text{cosec }\theta}$ and $\text{cosec }\theta = \frac{1}{\sin\theta}$

(ii) sec θ is the reciprocal ratio of cos θ.

$\cos\theta \times \sec\theta = \frac{b}{h} \times \frac{h}{b} = 1.$

$\therefore \cos\theta \times \sec\theta = 1 \Rightarrow \cos\theta = \frac{1}{\sec\theta}$ and $\sec\theta = \frac{1}{\cos\theta}$

(iii) cotθ is the reciprocal ratio of tan θ.

$\tan\theta \times \cot\theta = \frac{p}{b} \times \frac{b}{p} = 1.$

$\therefore$ $\tan\theta \times \cot\theta = 1 \Rightarrow \tan\theta = \frac{1}{\cot\theta}$ and $\tan\theta = \frac{1}{\cot\theta}$

B. Quotient Relations

(i) We have,

$\tan\theta = \frac{p}{b} = \frac{\frac{p}{h}}{\frac{b}{h}}$ [∵ Dividing numerator and denominator by h]

$= \frac{\sin\theta}{\cos\theta}$ [∵ By the definition of trigonometric ratios]

$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \sin\theta = \tan\theta \times \cos\theta$ and $\cos\theta = \frac{\sin\theta}{\tan\theta}$

(ii) We have,

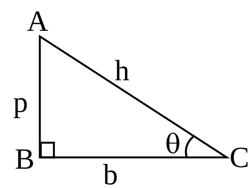
$\cot\theta = \frac{b}{p} = \frac{\frac{b}{h}}{\frac{p}{h}}$ [∵ Dividing numerator and denominator by h]

$= \frac{\cos\theta}{\sin\theta}$ [∵ By the definition of trigonometric ratios]

$\therefore \cot\theta = \frac{\cos\theta}{\sin\theta} \Rightarrow \sin\theta = \frac{\cos\theta}{\cot\theta}$ and $\cos\theta = \sin\theta \times \cot\theta$

C. Pythagorean Relation

We know that the Pythagoras theorem as $h^2 = p^2 + b^2$.



Now,

(i) $p^2 + b^2 = h^2$

Dividing on both sides by h^2, we get

$$\frac{p^2}{h^2} + \frac{b^2}{h^2} = \frac{h^2}{h^2}$$

or, $\left(\frac{p}{h}\right)^2 + \left(\frac{b}{h}\right)^2 = 1$

But we know that $\sin\theta = \frac{p}{h}$ and $\cos\theta = \frac{b}{h}$.

$\therefore$ **$\sin^2\theta + \cos^2\theta = 1$**

We derive, $\sin^2\theta = 1 - \cos^2\theta$

$\sin\theta = \sqrt{1 - \cos^2\theta}$ and

$\cos^2\theta = 1 - \sin^2\theta$

$\cos\theta = \sqrt{1 - \sin^2\theta}$.

(ii) $h^2 = p^2 + b^2$

$h^2 - p^2 = b^2$

Dividing on both sides by b^2, we get

$$\frac{h^2}{b^2} - \frac{p^2}{b^2} = \frac{b^2}{b^2}$$

or, $\left(\frac{h}{b}\right)^2 - \left(\frac{p}{b}\right)^2 = 1$

We have, $\sec\theta = \frac{h}{b}$ and $\tan\theta = \frac{p}{b}$.

$\therefore$ **$\sec^2\theta - \tan^2\theta = 1$**

We derive, $\sec^2\theta = 1 + \tan^2\theta$

$\sec\theta = \sqrt{1 - \tan^2\theta}$ and

$\tan^2\theta = \sec^2\theta - 1$

$\tan\theta = \sqrt{\sec^2\theta - 1}$.

(iii) $h^2 = p^2 + b^2$

$h^2 - b^2 = p^2$

Dividing on both sides by p^2, we get

$$\frac{h^2}{p^2} - \frac{b^2}{p^2} = \frac{p^2}{p^2}$$

or, $(\frac{h}{p})^2 - (\frac{b}{p})^2 = 1$

We have, $\text{cosec}\theta = \frac{h}{p}$ and $\cot\theta = \frac{b}{p}$

$\therefore$ **$\text{cosec}^2\theta - \cot^2\theta = 1$**

We derive, $\text{cosec}^2\theta = 1 + \cot^2\theta$

$\text{cosec}\theta = \sqrt{1 - \cot^2\theta}$ and

$\cot^2\theta = \text{cosec}^2\theta - 1$

$\cot\theta = \sqrt{\text{cosec}^2\theta - 1}$.

Example 1

Prove that: $\cot A \times \sin A = \cos A$

Solution: Here,

LHS $= \cot A \times \sin A$

$= \frac{\cos A}{\sin A} \times \sin A$

$= \cos A =$ RHS.

Example 2

Prove that: $\cos^2 x \times \text{cosec } x \times \tan^2 x = \sin x$

Solution: Here,

$$\text{LHS} = \cos^2 x \times \frac{1}{\sin x} \times \frac{\sin^2 x}{\cos^2 x} = \sin x = \text{RHS.}$$

Example 3

Prove that: $\cot A + \tan A = \dfrac{1}{\sin A \cos A}$

Solution: Here,

$$\text{LHS} = \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}$$

$$= \frac{\cos^2 A + \sin^2 A}{\sin A \cos A}$$

$$= \frac{1}{\sin A \cos A} \qquad [\because \sin^2 A + \cos^2 A = 1]$$

Example 4

Prove that: $\cot^2 x = (1 - \sin^2 x) \text{ cosec}^2 x$

Solution: Here,

$$\text{RHS} = (1 - \sin^2 x) \text{ cosec}^2 x$$

$$= \cos^2 x \times \frac{1}{\sin^2 x}$$

$$= \frac{\cos^2 x}{\sin^2 x}$$

$$= \cot^2 x = \text{RHS.}$$

Example 5

Prove that: $(\sin B - \cos B)^2 = 1 - 2 \sin B.\cos B$

Solution: Here,

$$\text{LHS} = (\sin B - \cos B)^2$$

$$= \sin^2 B - 2\sin B.\cos B + \cos^2 B$$

$= \sin^2 B + \cos^2 B - 2 \sin B.\cos B$

$= 1 - 2\sin B.\cos B = \text{RHS}.$

Example 6

Prove that: $\dfrac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

Solution: Here,

$$\text{LHS} = \frac{1}{\sec\theta - \tan\theta}$$

$$= \frac{\sec^2\theta - \tan^2\theta}{\sec\theta - \tan\theta} \qquad [\because \sec^2\theta - \tan^2\theta = 1]$$

$$= \frac{(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}{(\sec\theta - \tan\theta)}$$

$= \sec\theta + \tan\theta = \text{RHS}.$

Example 7

Prove that: $\tan^2\alpha + \cot^2\alpha + 2 = \sec^2\alpha \,\text{cosec}^2\alpha$

Solution: Here,

$\text{LHS} = \tan^2\alpha + \cot^2\alpha + 2$

$= (\tan^2\alpha + 1) + (\cot^2\alpha + 1)$

$= \sec^2\alpha + \text{cosec}^2\alpha$

$$= \frac{1}{\cos^2\alpha} + \frac{1}{\sin^2\alpha}$$

$$= \frac{\sin^2\alpha + \cos^2\alpha}{\cos^2\alpha \sin^2\alpha}$$

$$= \frac{1}{\cos^2\alpha \sin^2\alpha}$$

$= \sec^2\alpha \,\text{cosec}^2\alpha = \text{RHS}.$

Example 8

Prove that: $\dfrac{1}{1 - \cos\gamma} + \dfrac{1}{1 + \cos\gamma} = 2\,\text{cosec}^2\gamma$

Solution: Here,

$$\text{LHS} = \frac{1}{1 - \cos\gamma} + \frac{1}{1 + \cos\gamma}$$

$$= \frac{1 + \cos\gamma + 1 - \cos\gamma}{(1 - \cos\gamma)(1 + \cos\gamma)}$$

$$= \frac{2}{1 - \cos^2\gamma}$$

$$= \frac{2}{\sin^2\gamma}$$

$$= 2\,\text{cosec}^2\gamma = \text{RHS}.$$

Example 9

Prove that: $\sqrt{\frac{1 - \sin A}{1 + \sin A}} = \sec A - \tan A$

Solution: Here,

$$\text{LHS} = \sqrt{\frac{1 - \sin A}{1 + \sin A} \times \frac{1 - \sin A}{1 - \sin A}}$$

$$= \sqrt{\frac{(1 - \sin A)^2}{1 - \sin^2 A}}$$

$$= \sqrt{\frac{(1 - \sin A)^2}{\cos^2 A}}$$

$$= \frac{1 - \sin A}{\cos A}$$

$$= \frac{1}{\cos A} - \frac{\sin A}{\cos A}$$

$$= \sec A - \tan A = \text{RHS}.$$

Example 10

Prove that: $\sin^2 x.\cos^2 y - \cos^2 x.\sin^2 y = \sin^2 x - \sin^2 y$

Solution: Here,

$$\text{LHS} = \sin^2 x.\cos^2 y - \cos^2 x.\sin^2 y$$

$= \sin^2 x(1 - \sin^2 y) - (1 - \sin^2 x) \sin^2 y$

$= \sin^2 x - \sin^2 x.\sin^2 y - \sin^2 y + \sin^2 x.\sin^2 y$

$= \sin^2 x - \sin^2 y$ = RHS.

Example 11

Prove that: $\frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} = 2 \operatorname{cosec} A$

Solution: Here,

$$\text{LHS} = \frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A}$$

$$= \frac{\sin^2 A + (1 + \cos A)^2}{\sin A\,(1 + \cos A)}$$

$$= \frac{\sin^2 A + 1 + 2\cos A + \cos^2 A}{\sin A\,(1 + \cos A)}$$

$$= \frac{1 + 1 + 2\cos A}{\sin A\,(1 + \cos A)}$$

$$= \frac{2 + 2\cos A}{\sin A\,(1 + \cos A)}$$

$$= \frac{2\,(1 + \cos A)}{\sin A\,(1 + \cos A)}$$

$$= \frac{2}{\sin A}$$

$= 2 \operatorname{cosec} A$ = RHS.

Example 12

Prove that: $\frac{\tan^2 A}{\tan A - 1} + \frac{\cot A}{1 - \tan A} = 1 + \sec A.\operatorname{cosec} A$

Solution: Here,

$$\text{LHS} = \frac{\tan^2 A}{\tan A - 1} + \frac{\cot A}{1 - \tan A}$$

$$= \frac{\tan^2 A}{\tan A - 1} - \frac{1}{\tan A\,(\tan A - 1)}$$

$$= \frac{\tan^3 A - 1}{\tan A (\tan A - 1)}$$

$$= \frac{(\tan A - 1)(\tan^2 A + \tan A + 1)}{\tan A (\tan A - 1)} \quad [\because\ a^3 - b^3 = (a - b)(a^2 + ab + b^2]$$

$$= \frac{\tan^2 A + \tan A + 1}{\tan A}$$

$$= \frac{\tan^2 A}{\tan A} + \frac{\tan A}{\tan A} + \frac{1}{\tan A}$$

$$= \tan A + 1 + \frac{1}{\tan A}$$

$$= 1 + \left(\tan A + \frac{1}{\tan A}\right)$$

$$= 1 + \frac{\tan^2 A + 1}{\tan A}$$

$$= 1 + \sec^2 A \times \frac{\cos A}{\sin A}$$

$$= 1 + \frac{1}{\cos^2 A} \times \frac{\cos A}{\sin A}$$

$$= 1 + \frac{1}{\cos A} \times \frac{1}{\sin A}$$

$$= 1 + \sec A.\text{cosec } A = \text{RHS}$$

Example 13

Prove that: $\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{1 + \sin A}{\cos A}$

Solution: Here,

$$\text{LHS} = \frac{\tan A + \sec A - 1}{\tan A - \sec A + 1}$$

$$= \frac{(\tan A + \sec A) - (\sec^2 A - \tan^2 A)}{\tan A - \sec A + 1} \quad [\because \sec^2 A - \tan^2 A = 1]$$

$$= \frac{(\tan A + \sec A) + (\tan^2 A - \sec^2 A)}{\tan A - \sec A + 1}$$

$$= \frac{(\tan A + \sec A)(1 + \tan A - \sec A)}{\tan A - \sec A + 1} \qquad [\because a^2 - b^2 = (a-b)(a+b)]$$

$$= \tan A + \sec A$$

$$= \frac{\sin A}{\cos A} + \frac{1}{\cos A}$$

$$= \frac{1 + \sin A}{\cos A} = \text{RHS}$$

Exercise 5.4

1. Answer the following:

(a) Define trigonometric identity.

(b) Write the relation between $\sin\theta$ and $\cos\theta$.

(c) Write $\sec\theta$ in terms of $\tan\theta$.

(d) What is the product of $(\text{cosec}\,\theta + \cot\theta)$ and $(\text{cosec}\,\theta - \cot\theta)$?

2. Multiply:

(a) $(\sin A + \sin B)(\sin A - \sin B)$ (b) $(1 - \cos\alpha) - (1 + \cos\alpha)$

(c) $(1 + \cos x)(1 - \cos x)$ (d) $(1 + \tan^2 A)(1 - \tan^2 A)$

(e) $(1 + \sin\theta)(1 - \sin\theta)(1 + \sin^2\theta)$ (f) $(1 + \tan\alpha)(1 - \tan\alpha)(1 + \tan^2\alpha)$

3. Factorize:

(a) $\tan^2 A - \sin^2 A$ (b) $\cos^2 A - \sec^2 A$

(c) $\sin^2 x + \cos^2 x.\sin^2 x$ (d) $\sin^3\alpha - \cos^3\alpha$

(e) $\sec^4\theta - \cos^4\theta$ (f) $\sin^2 x + 5\sin x + 6$

4. Prove that:

(a) $\tan A \times \cos A = \sin A$ (b) $\cos\theta \times \text{cosec}\,\theta = \cot\theta$

(c) $\sec\alpha \times \sin\alpha \times \cot\alpha = 1$ (d) $\cot\beta \times \sin\beta = \cos\beta$

(e) $\dfrac{\sin A \times \text{cosec} A}{\tan A} = \cot A$ (f) $\dfrac{\cot A \times \tan A}{\sin A \times \cos A} = \text{cosec}\,\alpha \times \sec\alpha$

5. Prove that:

(a) $\cos^2 A - \cos^2 A \times \sin^2 A = \cos^4 A$ (b) $(1 - \cos^2\theta)(1 + \tan^2\theta) = \tan^2\theta$

(c) $\sin^2 A \times \cos^2 A + \sin^4 A = \sin^2 A$ (d) $(\cot^2\alpha + 1) \times \tan^2\alpha = \sec^2\alpha$

(e) $(1 + \sin A)^2 - (1 - \sin A)^2 = 4\sin A$

(f) $(1 + \tan \alpha)^2 + (1 - \tan \alpha)^2 = 2\sec^2 \alpha$

(g) $\dfrac{\operatorname{cosec}^2\theta - 1}{\operatorname{cosec}^2\theta} = \cos^2 \theta$

(h) $\tan^2 \alpha - \sin^2 \alpha = \tan^2 \alpha \times \sin^2 \alpha$

(i) $\cos \theta \sqrt{1 + \cot^2 \theta} = \sqrt{\operatorname{cosec}^2 \theta - 1}$

(j) $\sqrt{1 + \tan^2\alpha} \times \sqrt{1 - \cos^2\alpha} = \tan \alpha$

6. Prove that:

(a) $\dfrac{1 - \sin^4 A}{\cos^4 A} = 1 + 2\tan^2 A$

(b) $\dfrac{1 - \tan^4 A}{\sec^4 A} = 1 - 2\sin^2 A$

(c) $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

(d) $\dfrac{\sin^3\alpha + \cos^3\alpha}{\sin\alpha + \cos\alpha} = 1 - \sin\alpha.\cos\alpha$

(e) $\dfrac{\sin^3 A - \cos^3 A}{1 - \sin A \times \cos A} = \sin\alpha - \cos\alpha$

(f) $\dfrac{\tan A - 1}{\tan A + 1} = \dfrac{2\sin^2 A - 1}{1 + 2\sin A \times \cos A}$

(g) $\dfrac{1}{\sec\theta + \tan\theta} = \dfrac{1 - \sin\theta}{\cos\theta}$

(h) $\dfrac{1}{\operatorname{cosec} \theta + \cot \theta} = \dfrac{1 - \cos\theta}{\sin\theta}$

(h) $\dfrac{1}{\tan \theta + \cot \theta} = \sin\theta \times \cos\theta$

(i) $\dfrac{1}{\sec \theta - \tan \theta} = \dfrac{1 + \sin \theta}{\cos \theta} = \sec\theta + \tan\theta$

(j) $\dfrac{1}{\sec A - \tan A} - \dfrac{1}{\cos A} = \dfrac{1}{\cos A} - \dfrac{1}{\sec A + \tan A}$

(k) $\dfrac{1 - \sec A + \tan A}{1 + \sec A - \tan A} = \dfrac{\sec A + \tan A - 1}{\sec A + \tan A + 1}$

(l) $\dfrac{\cot A + \operatorname{cosec} A - 1}{\cot A + \operatorname{cosec} A + 1} = \dfrac{1 - \operatorname{cosec} A + \cot A}{1 + \operatorname{cosec} A - \cot A}$

(m) $(1 + \sin A + \cos A)^2 = 2(1 + \sin A)(1 + \cos A)$

(n) $(1 - \sin \alpha - \cos \alpha)^2 = 2(1 - \sin \alpha)(1 - \cos \alpha)$

(o) $\sin^2 x \times \sec^2 x + \tan^2 x \times \cos^2 x = \sin^2 x + \tan^2 x$

(p) $\dfrac{\tan x}{\sec x - 1} - \dfrac{\sin x}{1 + \cos x} = 2 \cot x$

(q) $\dfrac{\cos A - \sin A + 1}{\cos A + \sin A - 1} = \dfrac{1 + \cos A}{\sin A}$

(r) $\dfrac{1}{\cot A (1 - \cot A)} + \dfrac{1}{\tan A (1 - \tan A)} = 1 + \sec A \operatorname{cosec} A$

(s) $\dfrac{\sec A - \tan A}{\sec A + \tan A} = 1 - 2 \sec A \tan A + 2 \tan^2 A$

(t) $\dfrac{\cos A - \sin A + 1}{\cos A + \sin A + 1} = \dfrac{1 - \sin A}{\cos A}$

(u) $\dfrac{\sin\theta + \cos\theta + 1}{\sin\theta + \cos\theta - 1} - \dfrac{1 + \sin\theta - \cos\theta}{1 - \sin\theta + \cos\theta} = 2(1 + \operatorname{cosec}\theta)$

(v) $\operatorname{cosec}^4 A (1 - \cos^4 A) = 1 + 2 \cot^2 A$

(w) $(3 - 4 \sin^2 x)(\sec^2 x - 4 \tan^2 x) = (3 - \tan^2 x)(1 - 4 \sin^2 x)$

(x) $(\sec A + \operatorname{cosec} A)^2 = (1 + \cot A)^2 + (1 + \tan A)^2$

7. Prove that:

(a) $\sqrt{\dfrac{\sec\theta + 1}{\sec\theta - 1}} = \dfrac{1 + \cos\theta}{\sin\theta}$

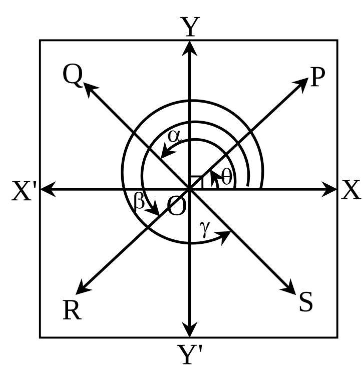
(b) $\sqrt{\dfrac{1 + \sin\theta}{1 - \sin\theta}} = \sec A + \tan A$

(c) $\sqrt{\dfrac{1 - \cos A}{1 + \cos A}} = \operatorname{cosec} A - \cot A$

(d) $\sqrt{\dfrac{1 + \sin A}{1 - \sin A}} - \sqrt{\dfrac{1 - \sin A}{1 + \sin A}} = 2 \tan A$

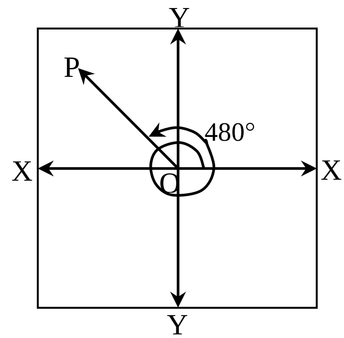
5.5 Trigonometric Ratios of Any Angle

Angles in Quadrant of cartesian plane



In the adjoining figure, two lines XX' and YY' are perpendicularly intersected at O, called origin and the line XOX' is called x-axis and YOY', y-axis. The revolving line OX revolves in anti-clockwise direction and it makes θ, α, β and γ in the first, second, third and forth quadrant at the position of OP, OQ, OR and OS respectively. The angle θ lies between 0° to 90°, i.e., $0° < \theta < 90°$. The angle α lies between 90° to 180°, i.e., $90° < \alpha < 180°$. The angle β lies between 180° to 270°, i.e., $180° < \beta < 270°$. The angle γ lies between 270° to 360°, i.e., $270° < \gamma < 360°$.

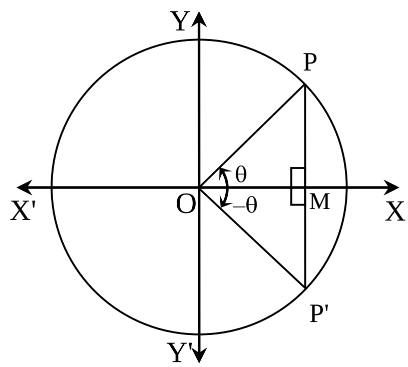
In which quadrant lies the angle 480°?



When 480° is divided by 90°, the quotient is 5 and the remainder will be 30°, i.e., 480° = 5 × 90° + 30°. So, the position of the revolving line OP lies 5 times 90° and more by 30°. Hence it lies in second quadrant.

By the next method, when 480° is divided by 360°, i.e., 480° = 1 × 360° + 120°, the remainder is 120° that lies in the second quadrant.

Trigonometric ratios of negative angle (– θ)



In the given figure, the revolving line OX revolves in anti-clockwise direction and makes an angle θ with x-axis. Take a point P on the revolving line and draw PM⊥OX and produce it such that PM = P'M. Join O and P'. Then ΔOMP and ΔOMP' are congruent by ASA condition of congruecy. Then,

OP = OP' and MP = – MP' or, MP' = – MP

Now, in right-angled triangle ΔOMP, we already know that

$$\sin\theta = \frac{p}{h} = \frac{MP}{OP},\ \cos\theta = \frac{b}{h} = \frac{OM}{OP}$$

$$\tan\theta = \frac{p}{b} = \frac{MP}{OM},\ \cot\theta = \frac{b}{p} = \frac{OM}{PM}$$

$$\sec\theta = \frac{h}{b} = \frac{OP}{OM},\ \mathrm{cosec}\theta = \frac{h}{p} = \frac{OP}{PM}$$

Again, in right angle ΔOMP',

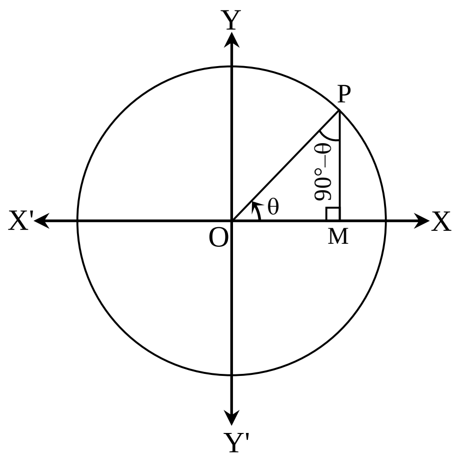
$$\sin(-\theta) = \frac{p}{h} = \frac{MP'}{OP'} = \frac{-MP}{OP} = -\sin\theta,\ \cos(-\theta) = \frac{b}{h} = \frac{OM}{OP'} = \frac{OM}{OP} = \frac{b}{h} = \cos\theta$$

$$\tan(-\theta) = \frac{p}{b} = \frac{MP'}{OM} = \frac{-MP}{OM} = -\tan\theta,\ \cot(-\theta) = \frac{b}{p} = \frac{OM}{MP'} = \frac{OM}{-MP} = -\cot\theta$$

$$\sec(-\theta) = \frac{h}{b} = \frac{OP'}{OM} = \frac{OP}{OM} = \sec\theta,\ \mathrm{cosec}(-\theta) = \frac{h}{p} = \frac{OP'}{MP'} = \frac{OP}{-MP} = -\mathrm{cosec}\theta$$

Trigonometric ratios of (90° – θ)

In the given figure, the revolving line OX revolves in anti-clockwise direction and makes an angle θ with x-axis. Take a point P on the revolving line and draw PM⊥OX. Therefore, ∠OPM = 90° – θ.



Now, in right-angled triangle OMP

$\sin\theta = \frac{MP}{OP}$, $\cos\theta = \frac{MP}{OP}$

$\tan\theta = \frac{MP}{OM}$, $\cot\theta = \frac{OM}{MP}$

$\sec\theta = \frac{OP}{OM}$, $\mathrm{cosec}\theta = \frac{OP}{MP}$

Now, in right-angled triangle OMP,

$\sin(90°–\theta) = \frac{p}{h} = \frac{OM}{OP}$, $\cos(90° – \theta) = \frac{b}{h} = \frac{MP}{OP}$

$\tan(90° – \theta) = \frac{p}{b} = \frac{OM}{MP}$, $\cot(90° – \theta) = \frac{b}{p} = \frac{MP}{OM}$

$\sec(90° – \theta) = \frac{h}{b} = \frac{OP}{MP}$, $\mathrm{cosec}(90° – \theta) = \frac{h}{p} = \frac{OP}{OM}$

Therefore, sin(90°–θ) = cosθ, cos(90° – θ) = sinθ, tan(90° – θ) = cotθ,

cot(90° – θ) = tanθ, sec(90° – θ) = cosecθ, cosec(90° – θ) = secθ

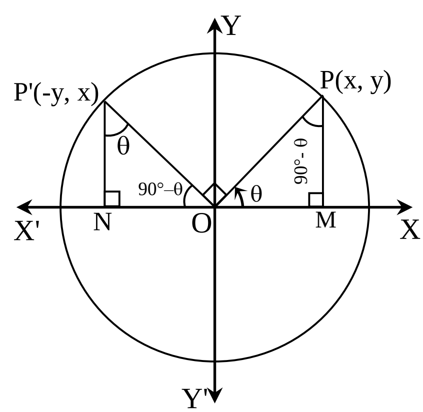
Trigonometric ratios of (90° + θ)

Let the revolving line OP makes an angle θ with X-axis in anti-clockwise direction in which OP = radius (r) and coordinates of P is (x, y).

∴ OM = x, PM = y and OP = r

The revolving line OP further revolves in the same direction and makes 90° with the position of P'. Now, draw PM⊥OX and P'N⊥OX'. Then, ∠POM = θ and ∠POP' = 90°.

∴ ∠MOP' = 90° + θ.



Again, in right-angled triangles PMO and P'NO,

∠MPO = ∠NOP' = 90° – θ, ∠POM = ∠NP'O = θ and OP = OP'.

By ASA condition of congruency of two triangles, ΔPMO ≅ ΔP'NO.

Then, ON = – PM = –y, P'N = OM = x [Since directions is along X']

Now,

$\sin\theta = \frac{\text{y coordinates of P}}{\text{radius}} = \frac{y}{r}$

$\cos\theta = \frac{\text{c coordinates of P}}{\text{radius}} = \frac{x}{r}$

$\tan\theta = \frac{\text{y coordinates of P}}{\text{x coordinates of P}} = \frac{y}{x}$

$\cot\theta = \frac{\text{x coordinates of P}}{\text{y coordinates of P}} = \frac{x}{y}$

$\sec\theta = \frac{\text{radius}}{\text{x coordinates of P}} = \frac{r}{x}$

$\text{cosec}\theta = \frac{\text{radius}}{\text{y coordinates of P}} = \frac{r}{y}$

Similarly,

$\sin(90° + \theta) = \frac{\text{y coordinates of P}'}{\text{radius}} = \frac{x}{r} = \cos\theta$

$\cos(90° + \theta) = \frac{\text{c coordinates of P}'}{\text{radius}} = \frac{-y}{r} = -\sin\theta$

$\tan(90° + \theta) = \frac{\text{y coordinates of P}'}{\text{x coordinates of P}'} = \frac{-x}{y} = -\cot\theta$

$\cot(90° + \theta) = \frac{\text{x coordinates of P}'}{\text{y coordinates of P}'} = \frac{-y}{x} = -\tan\theta$

$\sec(90° + \theta) = \frac{\text{radius}}{\text{x coordinates of P}'} = \frac{-r}{y} = -\text{cosec}\theta$

$\text{cosec}(90° + \theta) = \frac{\text{radius}}{\text{y coordinates of P}'} = \frac{r}{x} = \sec\theta$

Therefore,

sin(90° + θ) = cosθ

cos(90° + θ) = –sinθ

tan(90° + θ) = –cotθ

cosec(90° + θ) = secθ

sec(90° + θ) = –cosecθ

cot(90° + θ) = –tanθ

Trigonometric ratios of (180° – θ)

Here, $\sin(180° - \theta) = \sin[90° + (90° - \theta)] = \cos(90° - \theta) = \sin\theta$

$\cos(180° - \theta) = \cos[90° + (90° - \theta)] = -\sin(90° - \theta) = -\cos\theta$

$\tan(180° - \theta) = \tan[90° + (90° - \theta)] = -\cot(90° - \theta) = -\tan\theta$

$\cot(180° - \theta) = \cot[90° + (90° - \theta)] = -\tan(90° - \theta) = -\cot\theta$

$\sec(180° - \theta) = \sec[90° + (90° - \theta)] = -\text{cosec}(90° - \theta) = -\sec\theta$

$\text{cosec}(180° - \theta) = \text{cosec}[90° + (90° - \theta)] = \sec(90° - \theta) = \text{cosec}\,\theta$

Trigonometric ratios of (180° + θ)

Here, $\sin(180° + \theta) = \sin[90° + (90° + \theta)] = \cos(90° + \theta) = -\sin\theta$

$\cos(180° + \theta) = \cos[90° + (90° + \theta)] = -\sin(90° + \theta) = -\cos\theta$

$\tan(180° + \theta) = \tan[90° + (90° + \theta)] = -\cot(90° + \theta) = -(-\tan\theta) = \tan\theta$

$\cot(180° + \theta) = \cot[90° + (90° + \theta)] = -\tan(90° + \theta) = -(-\cot\theta) = \cot\theta$

$\sec(180° + \theta) = \sec[90° + (90° + \theta)] = -\text{cosec}(90° + \theta) = -\sec\theta$

$\text{cosec}(180° + \theta) = \text{cosec}[90° + (90° + \theta)] = \sec(90° + \theta) = -\text{cosec}\,\theta$

Trigonometric ratios of (270° – θ)

Here, $\sin(270° - \theta) = \sin[180° + (90° - \theta)] = -\sin(90° - \theta) = -\cos\theta$

$\cos(270° - \theta) = \cos[180° + (90° - \theta)] = -\cos(90° - \theta) = -\sin\theta$

$\tan(270° - \theta) = \tan[180° + (90° - \theta)] = \tan(90° - \theta) = \cot\theta$

$\cot(270° - \theta) = \cot[180° + (90° - \theta)] = \cot(90° - \theta) = \tan\theta$

$\sec(270° - \theta) = \sec[180° + (90° - \theta)] = -\sec(90° - \theta) = -\text{cosec}\,\theta$

$\text{cosec}(270° - \theta) = \text{cosec}[180° + (90° - \theta)] = -\text{cosec}(90° - \theta) = -\sec\theta$

Trigonometric ratios of (270° + θ)

Here, $\sin(270° + \theta) = \sin[180° + (90° + \theta)] = -\sin(90° + \theta) = -\cos\theta$

$\cos(270° + \theta) = \cos[180° + (90° + \theta)] = -\cos(90° + \theta) = -(-\sin\theta) = \sin\theta$

$\tan(270° + \theta) = \tan[180° + (90° + \theta)] = \tan(90° + \theta) = -\cot\theta$

$\cot(270° + \theta) = \cot[180° + (90° + \theta)] = \cot(90° + \theta) = -\tan\theta$

$\sec(270°+\theta) = \sec[180° + (90° + \theta)] = -\sec(90° + \theta) = -(-\text{cosec}\,\theta) = \text{cosec}\,\theta$

$\text{cosec}(270° + \theta) = \text{cosec}[180° + (90° + \theta)] = -\text{cosec}(90° + \theta) = -\sec\theta$

Trigonometric ratios of (360° – θ)

Here, sin(360° – θ) = sin[180° + (180° – θ)] = – sin(180° – θ) = – sin θ

cos(360° – θ) = cos [180° + (180° – θ)] = – cos(180° – θ) = – (–cos θ) = cos θ

tan(360° – θ) = tan [180° + (180° – θ)] = tan(180° – θ) = – tan θ

cot(360° – θ) = cot [180° + (180° – θ)] = cot(180° – θ) = – cot θ

sec(360° – θ) = sec [180° + (180° – θ)] = – sec(180° – θ) = – (– sec θ) = sec θ

cosec(360° – θ) = cosec [180° + (180° – θ)] = – cosec(180° – θ) = – cosec θ

Trigonometric ratios of (360° + θ)

Since the angle (360° + θ) lies in the first quadrant, so the trigonometric ratios of (360° + θ) are the same as the trigonometric ratios of θ.

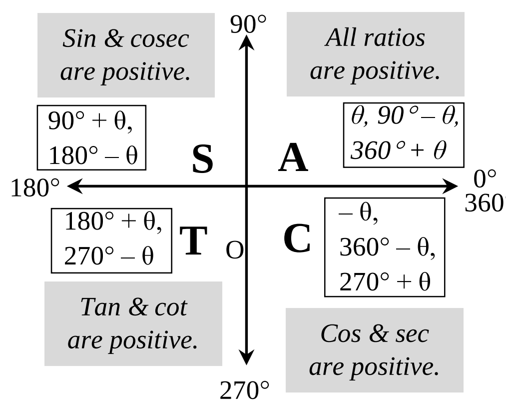
i.e., sin(360° + θ) = sin θ, cos(360° + θ) = cos θ

tan(360° + θ) = tan θ, cot(360° + θ) = cot θ

sec(360° + θ) = sec θ, cosec(360° + θ) = cosec θ

CAST Rule

Summarise of the trigonometric ratios of any angles as follow:



Note:

1) If n is an odd number in (n × 90° ± θ), then the trigonometric ratio will be changed.

2) If n is an even number or zero in (n × 90° ± θ), then the trigonometric ratio will not be changed.

Example 1

Find the value of sin 150°.cos 120° + $\sin^2$150° + $\cos^2$120°.

Solution: Here,

sin 150°.cos 120° + $\sin^2$150° + $\cos^2$120°

= sin(90° + 60°).cos(90° + 30°) + $\sin^2$(90° + 60°) + $\cos^2$(90° + 30°)

= cos 60°.(– sin 30°) + $\cos^2$ 60° + $\sin^2$ 30°

$$= \frac{1}{2} \times \left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= -\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{1}{4}$$

Example 2

Prove that: $\sin^2 \theta + \sin^2 (90° - \theta) = \cos^2 \theta + \cos^2 (90° - \theta)$

Solution: Here,

LHS $= \sin^2 \theta + \sin^2 (90° - \theta)$

$= \sin^2 \theta + \cos^2 \theta \qquad [\because \sin (90° - \theta) = \cos \theta]$

$= 1$

RHS $= \cos^2 \theta + \cos^2 (90° - \theta)$

$= \cos^2 \theta + \sin^2 \theta \qquad [\because \cos (90° - \theta) = \sin\theta]$

$= 1$

$\therefore$ LHS = RHS

Example 3

Prove that: $\tan x + \tan (180° - x) + \cot (90° + x) + \cot (90° - x) = 0$

Solution: Here,

LHS $= \tan x + \tan (180° - x) + \cot (90° + x) + \cot (90° - x)$

$= \tan x + (-\tan x) + (-\tan x) + \tan x$

$= 2 \tan x - \tan x - \tan x = 2 \tan x - 2\tan x = 0 =$ RHS.

Example 4

Prove that: $\dfrac{\tan (90° + \alpha) . \sec (270° - \alpha) . \sin(-\alpha)}{\cos (180° + \alpha) . \cos (-\alpha)} = \operatorname{cosec} \alpha.\sec \alpha$

Solution: Here,

LHS $= \dfrac{\tan (90° + \alpha) . \sec (270° - \alpha) . \sin(-\alpha)}{\cos (180° + \alpha) . \cos (-\alpha)}$

$= \dfrac{-\cot \alpha (-\operatorname{cosec} \alpha) (-\sin \alpha)}{-\cos\alpha.\cos\alpha}$

$$= \frac{-\cot\alpha.\text{cosec}\,\alpha.\sin\alpha}{-\cos^2\alpha}$$

$$= \frac{\cos\alpha}{\text{Sin}\,\alpha\,.\cos^2\alpha}$$

$$= \frac{1}{\sin\alpha.\cos\alpha} = \text{cosec}\,\alpha.\sec\alpha = \text{RHS.}$$

Example 5

Prove that: sin 120° – cos 150° + tan 135° = $\sqrt{3}$ – 1

Solution: Here,

LHS = sin (180° – 60°) – cos (180° – 30°) + tan (180° – 45°)

= sin 60° – (– cos 30°) + (– tan 45°)

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - 1 = 2 \times \frac{\sqrt{3}}{2} - 1 = \sqrt{3} - 1 = \text{RHS.}$$

Example 6

Prove that: sin 112° + cos 74° – sin 68° + cos 106° = 0

Solution: Here,

LHS = sin 112° + cos 74° – sin 68° + cos 106°

= sin (180° – 68°) + cos 74° – sin 68° + cos (180° – 74°)

= sin 68° + cos 74° – sin 68° – cos 74°

= 0 = RHS.

Example 7

Find the value of x from cosec (90° + γ) + x cos γ.cot (90° + γ) = sin (90° + γ)

Solution: Here,

cosec (90° + γ) + x cos γ.cot (90° + γ) = sin (90° + γ)

or, sec γ + x cos γ (– tan γ) = cos γ

or, $\frac{1}{\cos\gamma} - x\cos\gamma\,.\frac{\sin\gamma}{\cos\gamma} = \cos\gamma$

or, $\frac{1}{\cos\gamma} - x\sin\gamma = \cos\gamma$

or, $-x \sin \gamma = \cos \gamma - \frac{1}{\cos \gamma}$

or, $-x \sin \gamma = \frac{\cos^2 \gamma - 1}{\cos \gamma}$

or, $x \sin \gamma . \cos \gamma = 1 - \cos^2 \gamma$

or, $x \sin \gamma . \cos \gamma = \sin^2 \gamma$

or, $x = \frac{\sin^2 \gamma}{\sin \gamma . \cos \gamma} = \frac{\sin \gamma}{\cos \gamma} = \tan \gamma$

Hence, $x = \tan \gamma$

Exercise 5.5

1. **Answer the following in single sentence:**
   (a) Which trigonometric ratio is equal to $\sin(-\theta)$?
   (b) Write the trigonometric ratio is equal to $\tan(180° + A)$.
   (c) Which trigonometric ratio is equal to $\cos(270° - A)$?
   (d) Which trigonometric ratio is equal to secant of $(180° - \theta)$?
2. **Determine the values of the following trigonometric ratios:**
   (a) $\sin 150°$ (b) $\cos 135°$ (c) $\cot 315°$ (d) $\cos 855°$
   (e) $\tan 1035°$ (f) $\cot 1755°$ (g) $\operatorname{cosec}(-1410°)$
3. **Simplify:**
   (a) $\sin(90° - A) \times \tan(180° + A) - \tan(-A) \times \sec(270° - A)$
   (b) $\cos(90° - A) \times \sin(360° + A) \times \tan(180° - A) \times \sec(270° + A) \times \operatorname{cosec}(90° + A)$
4. **Prove that:**
   (a) $\sec \alpha . \operatorname{cosec}(90° - \alpha) - \tan \alpha . \cot(90° - \alpha) = 1$
   (b) $\sin^2 \theta . \operatorname{cosec}^2(90° - \theta) - \tan^2 \theta . \sec^2(90° - \theta) = -1$
   (c) $\tan^2\left(\frac{\pi}{2} - P\right) . \sin P - \sin^2\left(\frac{\pi}{2} - P\right) . \operatorname{cosec} P = 0$
   (d) $\operatorname{cosec} A . \cos\left(\frac{\pi}{2} - A\right) - \sin A . \sec\left(\frac{\pi}{2} - A\right) = 0$

(e) $\cot \alpha + \cot\left(\frac{\pi}{2} - \alpha\right) = \text{cosec}\ \alpha . \text{cosec}\left(\frac{\pi}{2} - \alpha\right)$

(f) $\frac{\tan^2 \alpha}{\cos^2 (90° - \alpha)} - \frac{\sin^2 \alpha}{\sin^2 (90° - \alpha)} = 1$

5. Prove that:

(a) $\sin 420° \cos 390° + \cos (-300°) \sin (-330°) = 1$

(b) $\cos 120°. \sin 150° + \cos 330°. \sin 300° = -1$

(c) $\cos 240°. \sin 300° - \sin 330°. \cos 300° = \frac{1 + \sqrt{3}}{4}$

(d) $\cos 240°. \cos 120° - \sin 120°. \cos 150° = 1$

6. Show that:

(a) $\cos 25°.\cos 40° = \sin 65°.\sin 50°$

(b) $\sin155°.\cos 165° = -\sin 25°.\cos 15°$

(c) $\cos 12° + \text{cosec}\ 36° + \cot 72° = \sin 78° + \sec 54° + \tan 18°$

(d) $\tan 32° + \cot 53° - \text{cosec}\ 80° = \tan 37° + \cot 58° - \sec 10°$

(e) $\sin 81° + \sec 54° + \tan 18° = \cos 9° + \text{cosec}\ 36° + \cot 72°$

(f) $\sin 9°. \sin 27°. \sin 63°. \sin 81° = \cos 9°. \cos 27° \cos 63° \cos 81°$

(g) $\tan 9°. \tan 27°. \tan 45°. \tan 63° \tan 81° = 1$

7. Find the values of *x*:

(a) $\cos 150° + \sin 120° + \sin^2 150° + \cos^2 120°$

(b) $\cos^2 135° + \sin^2 150° - \sin^2 120° - \cot^2 120°$

(c) $\cos^2 90° + \cos^2 120° + \cos^2 135° + \cos^2 150° + \cos^2 180°$

(d) $\sin^2 120° - \cos^2 120° - \sin^2 135° - \tan^2 150°$

(e) $2 \cos^2 45° + \sin 30° + \frac{1}{2} \cos 180° - \tan 45°$

(f) $\tan^2 45° - 4 \sin^2 60° + 2 \cos^2 45° + \sec^2 180° + \text{coec}\ 135°$

(g) $\sin^2 180° + \sin^2 150° + \sin^2 135° + \sin^2 120° + \sin^2 90°$

(h) $2\cos^2 135° + \sin 150° + \frac{1}{2}\cos 180° + \tan^2 135°$

(i) $\cos\frac{\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{5\pi}{8} + \cos\frac{7\pi}{8}$

(j) $\sin^2\left(\frac{7\pi}{8}\right) + \sin^2\left(\frac{5\pi}{8}\right) + \sin^2\left(\frac{3\pi}{8}\right) + \sin^2\left(\frac{\pi}{8}\right)$

8. Find the values of:

(a) $\frac{\sin(90° + A) \times \operatorname{cosec}(90° + A)}{\cos A \times \cot(90° + A)}$

(b) $\frac{\sin(90° + \theta).\cos(-\theta).\cot(180° - \theta)}{\cos(360° - \theta).\cos(180° + \theta).\tan(90° - \theta)}$

(c) $\frac{\tan(180° - A).\cot(90° - A)\cos(360° - A)}{\tan(180° - A)\tan(90° + A)\sin(-A)}$

(d) $\frac{\cos(90° - \alpha).\cot(90° - \alpha).\cos(180° - \alpha)}{\tan(180° - \alpha).\tan(90° - \alpha)\cos(90° + \alpha)}$

(e) $\frac{\sin(180° - \theta)}{\sin(90° + \theta)} \times \frac{\tan(90° + \theta)}{\cos(180° - \theta)} \times \frac{\sec(90° + \theta)}{\cot(180° - \theta)}$

(f) $\frac{\cos(270° - \alpha).\sec(180° - \alpha).\sin(270° + \alpha)}{\cos(90° + \alpha).\cos(180° - \alpha).\sin(180° + \alpha)}$

9. Find the value of:

(a) $2\cot 120° - x.\sin 120°.\cos 180° = \tan 150°$

(b) $2\cot 120° - \tan 150° = x.\sin 120°.\cos 180°$

(c) $\tan 225° - x.\sin 315°.\cos 135°.\tan^2 60° = \operatorname{cosec}^2 30°$

(d) $\tan^2 135° - \sec^2 60° = x.\sin 135°.\cos 45°.\tan 60°$

(e) $\tan(180° - A).\cot(90° + A) + x\cos(90° + A).\cos(90° - A) = \sin A.\sin(180° - A)$

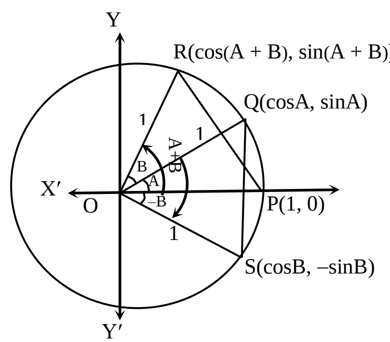
(f) $x.\cot\alpha.\tan(90° + \alpha) = \tan(90° + \alpha).\cot(180° - \alpha) + x\sec(90° + \alpha).\operatorname{cosec}\alpha$

5.6 Trigonometric Ratios of Compound Angles

The sum or difference of two or more than two angles is called a compound angle. Let A and B be two angles then A + B or A – B is called the compound angle and the trigonometric ratios of A + B and A – B are denoted by sin(A + B), cos(A + B) etc.

Trigonometric Ratio for cosine of angles (A + B) and (A – B)

Let OP be a revolving line with a unit length, starting from OX, describing $\angle POQ = A$, $\angle QOR = B$ and $\angle POS = -B$. Then $\angle POR = A + B$. So, the coordinates of P, Q, R and S are (1, 0), (cos A, sin A), (cos(A + B), sin(A + B)) and (cos B, –sin B) respectively. Since PR and QS are the opposite side of the same magnitude of angles (A + B) and [A + (–B)] respectively, PR = QS.



Now, we have

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{or, } PR = \sqrt{(\cos(A+B) - 1)^2 + (\sin(A+B) - 0)^2}$$

$$= \sqrt{\cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B)}$$

$$= \sqrt{1 - 2\cos(A+B) + 1} = \sqrt{2 - 2\cos(A+B)}.$$

$$QS = \sqrt{(\cos B - \cos A)^2 + (-\sin B - \sin A)^2}$$

$$= \sqrt{\cos^2 B - 2\cos A.\cos B + \cos^2 A + \sin^2 B + 2\sin A.\sin B + \sin^2 A}$$

$$= \sqrt{1 + 1 - 2\cos A.\cos B + 2\sin A.\sin B}$$

$$= \sqrt{2 - 2(\cos A.\cos B - \sin A.\sin B}.$$

Since PR = QS, so

$$\sqrt{2 - 2\cos(A+B)} = \sqrt{2 - 2(\cos A.\cos B - \sin A.\sin B}$$

or, $2 - 2\cos(A + B) = 2 - 2(\cos A.\cos B - \sin A.\sin B)$.

$\therefore \cos(A + B) = \cos A.\cos B - \sin A.\sin B.$

Taking B = – B, we get

$\cos[A + (-B)] = \cos A.\cos(-B) - \sin A.\sin(-B)$

$\therefore \cos(A - B) = \cos A.\cos B + \sin A.\sin B.$

Trigonometric Ratios for Sine Angles of (A + B) and (A – B)

We have,

$\sin(A+B) = \cos[90^o - (A + B)] = \cos[(90^o - A) - B]$

$= \cos(90^o - A).\cos B + \sin(90^o - A).\sin B$

$= \sin A.\cos B + \cos A.\sin B.$

$\therefore \sin(A+B) = \sin A.\cos B + \cos A.\sin B.$

Again, taking B = –B, we get

$\sin[A + (-B)] = \sin A.\cos(-B) + \cos A.\sin(-B)$

$\therefore \sin(A - B) = \sin A.\cos B - \cos A.\sin B.$

Trigonometric Ratios for Tangent Angles of (A + B) and (A – B):

We have,

$$\tan(A + B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A.\cos B + \cos A \cdot \sin B}{\cos A.\cos B - \sin A \cdot \sin B}$$

$$= \frac{\frac{\sin A \cdot \cos B}{\cos A \cdot \cos B} + \frac{\cos\ A \cdot \sin B}{\cos A \cdot \cos B}}{\frac{\cos A \cdot \cos B}{\cos A \cdot \cos B} - \frac{\sin A \cdot \sin B}{\cos A \cdot \cos B}} = \frac{\tan A + \tan B}{1 - \tan A.\tan B}$$

[$\because$ Diving by cosA·cosB on numerator and denominator.]

$$\therefore \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A.\tan B}$$

Again taking B = –B, we get

$$\therefore \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

Trigonometric Ratios for Cotangent Angles of (A + B) and (A – B)

We have,

$\cot(A + B) = \tan[90^o - (A + B)] = \tan[(90^o - A) - B]$

$$= \frac{\tan(90^o - A) - \tan B}{1 + \tan(90^o - A) \cdot \tan B} = \frac{\cot A - \frac{1}{\cot B}}{1 + \cot A \cdot \frac{1}{\cot B}} = \frac{\frac{\cot A.\cot B - 1}{\cot B}}{\frac{\cot B + \cot A}{\cot B}} = \frac{\cot A.\cot B - 1}{\cot B + \cot A}$$

$$\therefore \cot(A + B) = \frac{\cot A.\cot B - 1}{\cot B + \cot A}$$

Again, taking B = –B, we get

$$\cot[A + (-B)] = \frac{\cot A.\cot(-B) - 1}{\cot(-B) + \cot A}$$

$$\text{or, } \cot(A - B) = \frac{-\cot A.\cot B - 1}{-\cot B + \cot A}$$

$$\therefore \cot(A - B) = \frac{\cot A.\cot B + 1}{\cot B - \cot A}$$

Example 1

Find values of cos 15° without using scientific calculator and trigonometric table.

Solution: We have,

$$\cos 15° = \cos(45° - 30°)$$

$$= \cos 45°\cdot\cos 30° + \sin 45°\cdot\sin 30°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3} + 1}{2\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\therefore \cos 15° = \frac{\sqrt{3} + 1}{2\sqrt{2}} \text{ or } \frac{\sqrt{6} + \sqrt{2}}{4}$$

Example 2

Prove that: $\sin 105° + \cos 105° = \frac{1}{\sqrt{2}}$

Solution: Here,

$$\text{LHS} = \sin 105° + \cos 105°$$

$$= \sin(60° + 45°) + \cos(60° + 45°)$$

$$= \sin 60°\cdot\cos 45° + \cos 60°\cdot\sin 45° + \cos 60°\cdot\cos 45° - \sin 60°\cdot\sin 45°$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{1 + 1}{2\sqrt{2}} = \frac{2}{2\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} = \text{RHS.}$$

Example 3

If $\sin A = \frac{3}{5}$ and $\sin B = \frac{12}{13}$, find the values of $\sin (A + B)$ and $\tan (A + B)$.

Solution: Here,

$$\sin A = \frac{3}{5} \quad \sin B = \frac{12}{13}$$

$$\cos A = \sqrt{1 - \sin^2 A} \qquad \cos B = \sqrt{1 - \sin^2 B}$$

$$= \sqrt{1 - \left(\frac{3}{5}\right)^2} \qquad = \sqrt{1 - \left(\frac{12}{13}\right)^2}$$

$$= \sqrt{1 - \frac{9}{25}} \qquad = \sqrt{1 - \frac{144}{169}}$$

$$= \sqrt{\frac{25 - 9}{25}} \qquad = \sqrt{\frac{169 - 144}{169}}$$

$$= \sqrt{\frac{16}{25}} = \frac{4}{5} \qquad = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$\therefore \cos A = \frac{4}{5} \qquad \therefore \cos B = \frac{5}{13}$$

Now, we know that

$$\sin (A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

$$= \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13} = \frac{15}{65} + \frac{48}{65} = \frac{15 + 48}{65} = \frac{63}{65}$$

$$\therefore \sin (A + B) = \frac{63}{65}$$

Again,

$$\therefore \tan A = \frac{\sin A}{\cos A} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4} \qquad \therefore \tan B = \frac{\sin B}{\cos B} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{5}$$

Now, we have

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A.\tan B} = \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4} \times \frac{12}{5}} = \frac{\frac{15+48}{20}}{\frac{20-36}{20}} = -\frac{63}{16}$$

$$\therefore \tan(A+B) = -\frac{63}{16}$$

Example 4

If $\tan\alpha = \frac{5}{6}$ and $\tan\beta = \frac{1}{11}$, prove that: $\alpha + \beta = \frac{\pi^c}{4}$

Solution: Here,

$\tan\alpha = \frac{5}{6}$ and $\tan\beta = \frac{1}{11}$

Now, we know that

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha.\tan\beta}$$

$$= \frac{\frac{5}{6} + \frac{1}{11}}{1 - \frac{5}{6} \times \frac{1}{11}} = \frac{\frac{55+6}{66}}{\frac{66-5}{66}} = \frac{61}{61} = 1 = \tan 45° = \tan\frac{\pi^c}{4}$$

$$\therefore \quad (\alpha+\beta) = \frac{\pi^c}{4}.$$

Example 5

Prove that: $\sin(x+y) + \sin(x-y) = 2\sin x.\cos y$

Solution: Here,

LHS $= \sin(x+y) + \sin(x-y)$

$= \sin x\cdot\cos y + \cos x\cdot\sin y + \sin x\cdot\cos y - \cos x\cdot\sin y$

$= 2\sin x\cdot\cos y$ = RHS.

Example 6

Prove that: $\cot(A-B) = \frac{\cot B.\cot A + 1}{\cot B - \cot A}$

Solution: Here,

$$\text{LHS} = \cot(A - B)$$

$$= \frac{\cos(A - B)}{\sin(A - B)}$$

$$= \frac{\cos A.\cos B + \sin A.\sin B}{\sin A.\cos B - \cos A.\sin B}$$

$$= \frac{\dfrac{\cos A.\cos B}{\sin A.\sin B} + \dfrac{\sin A.\sin B}{\sin A.\sin B}}{\dfrac{\sin A.\cos B}{\sin A.\sin B} - \dfrac{\cos A.\sin B}{\sin A.\sin B}}$$

[$\therefore$ Dividing sinA·sinB on numerator and denominator]

$$= \frac{\cot A.\cot B + 1}{\cot B - \cot A} = \text{RHS.}$$

Example 7

Prove that: $\dfrac{\cos 10° - \sin 10°}{\cos 10° + \sin 10°} = \tan 35°$

Solution

$$\text{LHS} = \frac{\cos 10° - \sin 10°}{\cos 10° + \sin 10°}$$

$$= \frac{\cos(45° - 35°) - \sin(45° - 35°)}{\cos(45° - 35°) + \sin(45° - 35°)}$$

$$= \frac{\cos 45°.\cos 35° + \sin 45°.\ \sin 35° - \{\sin 45°.\cos 35° - \cos 45°.\sin 35°\}}{\cos 45°.\cos 35° + \sin 45°.\ \sin 35° + \{\sin 45°.\cos 35° - \cos 45°.\sin 35°\}}$$

$$= \frac{\frac{1}{\sqrt{2}}.\cos 35° + \frac{1}{\sqrt{2}}\sin 35° - \frac{1}{\sqrt{2}}.\cos 35° + \frac{1}{\sqrt{2}}\sin 35°}{\frac{1}{\sqrt{2}}.\cos 35° + \frac{1}{\sqrt{2}}\sin 35° + \frac{1}{\sqrt{2}}.\cos 35° - \frac{1}{\sqrt{2}}\sin 35°}$$

$$= \frac{2.\sqrt{2}.\sin 35°}{2.\sqrt{2}.\cos 35°}$$

$$= \tan 35° = \text{RHS.}$$

"Alternatively"

$$\text{RHS} = \tan 35°$$

$$= \tan(45° - 10°)$$

$$= \frac{\tan45° - \tan10°}{1 + \tan45°.\tan10°}$$

$$= \frac{1 - \frac{\sin10°}{\cos10°}}{1 + 1 \times \frac{\sin10°}{\cos10°}}$$

$$= \frac{\frac{\cos10° - \sin10°}{\cos10°}}{\frac{\cos10° + \sin10°}{\cos10°}}$$

$$= \frac{\cos10° - \sin10°}{\cos10° + \sin10°} = \text{RHS.}$$

Example 8

Prove that: 1 – tan30° – tan15° = tan30° . tan15°

Solution: Here,

LHS $= 1 - \tan30° - \tan15°$

$= 1 - (\tan30° + \tan15°)$

$= 1 - \frac{\tan30° + \tan15°}{1 - \tan30°.\tan15°} \times (1 - \tan30°.\tan15°)$

$= 1 - \tan(30° + 15°)(1 - \tan30°.\tan15°)$

$= 1 - \tan45°(1 - \tan30°.\tan15°)$

$= 1 - 1(1 - \tan30°.\tan15°)$

$= 1 - 1 + \tan30°.\tan15°$

$= \tan30°.\tan15° =$ RHS.

"Alternatively"

Here, we have

$45° = 30° + 15°$

or, $\tan 45° = \tan(30° + 15°)$ [∵ Taking tangent ratio on both sides]

or, $1 = \frac{\tan30° + \tan15°}{1 - \tan30° \tan15°}$

or, $1 - \tan30°.\tan15° = \tan30° + \tan15°$

or, $1 - \tan30° - \tan15° = \tan30°.\tan15°$

Example 9

If $(A + B) = \frac{\pi^C}{4}$, prove that: $(1 + \tan A)(1 + \tan B) = 2$

Solution: Here,

$$A + B = \frac{\pi^C}{4}$$

$\tan(A + B) = \tan\frac{\pi^C}{4}$ [∵ Taking tangent ratio on both sides]

or, $\frac{\tan A + \tan B}{1 - \tan A.\tan B} = \tan45°$

or, $\frac{\tan A + \tan B}{1 - \tan A.\tan B} = 1$

or, $\tan A + \tan B = 1 - \tan A.\tan B$

or, $\tan A + \tan B + \tan A.\tan B = 1$

or, $1 + \tan A + \tan B + \tan A.\tan B = 1 + 1$ [∵ *Adding 1 on both sides*]

or, $1(1 + \tan A) + \tan B(1 + \tan A) = 2$

∴ $(1 + \tan A)(1 + \tan B) = 2$

Example 10

If $\tan\alpha = k.\tan\beta$, verify that: $(k + 1).\sin(\alpha - \beta) = (k - 1).\sin(\alpha + \beta)$

Solution:

Given, $\tan\alpha = k.\tan\beta$.

To Prove: $(k + 1).\sin(\alpha - \beta) = (k - 1).\sin(\alpha + \beta)$

or, $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} = \frac{k-1}{k+1}$.

Now, LHS $= \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} = \frac{\sin\alpha.\cos\beta-\cos\alpha.\sin\beta}{\sin\alpha.\cos\beta+\cos\alpha.\sin\beta}$

$= \frac{\frac{\sin\alpha.\cos\beta}{\cos\alpha.\cos\beta}-\frac{\cos\alpha.\sin\beta}{\cos\alpha.\cos\beta}}{\frac{\sin\alpha.\cos\beta}{\cos\alpha.\cos\beta}+\frac{\cos\alpha.\sin\beta}{\cos\alpha.\cos\beta}}$ [∵ Dividing cosA.cosB on numerator and denominator]

$= \frac{\tan\alpha - \tan\beta}{\tan\alpha + \tan\beta}$

$= \frac{k.\tan\beta - \tan\beta}{k.\tan\beta + \tan\beta}$

$= \frac{\tan\beta(k-1)}{\tan\beta(k+1)}$

$= \frac{k-1}{k+1}$ = RHS.

"Alternatively"

Given, tan α = k. tan β

or, $\frac{\tan\alpha}{\tan\beta} = \frac{k}{1}$

Using componendo and dividendo rules, we have

or, $\frac{\tan\alpha - \tan\beta}{\tan\alpha + \tan\beta} = \frac{k-1}{k+1}$

or, $\frac{\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta}}{\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta}} = \frac{k-1}{k+1}$

or, $\frac{\frac{\sin\alpha.\cos\beta - \cos\alpha.\sin\beta}{\cos\alpha.\cos\beta}}{\frac{\sin\alpha.\cos\beta + \cos\alpha.\sin\beta}{\cos\alpha.\cos\beta}} = \frac{k-1}{k+1}$

or, $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} = \frac{k-1}{k+1}$

or, (k + 1) sin(α – β) = (k – 1) sin(α – β).

Exercise 5.6

1. Write the answer in one sentence:

(a) Write the formula of sin(A + B).

(b) Write the formula of cos(A + B).

(c) Write the formula of tan(A – B).

(d) Write the formula of cot(A + B).

2. Find the value of trigonometric ratios without using trigonometric table and scientific devices.

(a) cos 15^o (b) sin 75^o (c) sin 105^o (d) tan 15^o (e) tan 75^o

(f) $\tan 105^o$ (g) $\cos 195^o$ (h) $\sin 255^o$ (i) $\cot \frac{\pi^c}{6}$ (j) $\sin \frac{7\pi^c}{12}$

3. **Without using trigonometric table and scientific devices, calculate the following trigonometric ratios.**

(a) $\sin 15^o + \cos 75^o$ (b) $\sin 75^o + \sin 105^o$ (c) $\cos 15^o - \sin 75^o$

4. **Prove that:**

(a) $\sin 105° + \cos 105° = \frac{1}{\sqrt{2}}$ (b) $\sin 75° - \sin 15° = \frac{1}{\sqrt{2}}$

(c) $\sin 65^o - \sin 25^o = \sqrt{2}\sin 20^o$ (d) $\sqrt{3}\cos 20^o + \sin 20^o = 2\sin 80^o$

5. **Verify that:**

(a) $\sin(60^o - \theta).\cos(30^o + \theta) + \cos(60^o - \theta).\sin(30^o + \theta) = 1$

(b) $\cos(60^o - \alpha).\cos(30^o - \beta) - \sin(60^o - \alpha).\sin(30^o - \beta) = \sin(\alpha + \beta)$

(c) $\sin(2m + 1)A.\cos(2m - 1)A - \cos(2m + 1)A.\sin(2m + 1)A = \sin 2m$

(d) $\cos(120^o + x) + \cos x + \cos(120^o - x) = 0$

6. **Prove that:**

(a) $\frac{\tan 4A - \tan 3A}{1 + \tan 4A.\tan 3A} = \tan A$ (b) $\frac{\tan nx + \tan x}{1 - \tan nx.\tan x} = \tan(n + 1)x$

(c) $\frac{\cot B \cot A + 1}{\cot B - \cot A} = \cot(A - B)$ (d) $\tan(A + B).\tan(A - B) = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A.\tan^2 B}$

(e) $\frac{\tan(A + B) - \tan A}{1 + \tan(A + B).\tan A} = \tan B$ (f) $\frac{\tan(A + B + C) - \tan(A - B + C)}{1 + \tan(A + B + c).\tan(A - B + C)} = \tan 2B$

(g) $\tan(A + \frac{\pi^c}{4}) = \frac{\sin A - \cos A}{\sin A + \cos A}$ (h) $\tan\left(\frac{3\pi^c}{4} - A\right) = \frac{\tan A + 1}{\tan A - 1}$

(i) $\frac{\sin(A - B)}{\sin A.\sin B} = \cot B - \cot A$ (j) $\frac{\cos(A - B)}{\sin A.\sin B} = \cot A.\cot B + 1$

7. **Show that the followings:**

(a) $\tan 28^o + \tan 17^o + \tan 28^o.\tan 17^o = 1$

(b) $1 + \tan 61^o.\tan 16^o = \tan 61^o - \tan 16^o$

(c) $1 + \cot 20^o + \cot 25^o = \cot 20^o.\cot 25^o$

8. **If $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$, find the values of the following compound angles:**

(a) $\sin(A + B)$ (b) $\cos(A - B)$ (c) $\tan(A + B)$

9. If sin α = $\frac{1}{\sqrt{10}}$ and cos β = $\frac{2}{\sqrt{5}}$, find the values of the following:

(a) sin(α – β) (b) cos(α + β) (c) cot(α – β)

10. If tanα = $\frac{3}{4}$ and tanβ = $\frac{5}{12}$, prove that:

(a) tan(α + β) = $1\frac{23}{33}$ (b) sin(α – β) = $\frac{16}{65}$ (c) cot(α – β) = $3\frac{15}{16}$

11. (a) If tanα = $\frac{2}{3}$ and tanβ = $\frac{1}{5}$, show that: α + β = $\frac{\pi^c}{4}$.

(c) If tan α = m and tan β = $\frac{1}{m}$, show that: α + β = 90°.

(c) If cos x = $\frac{1}{7}$ and cosy = $\frac{13}{14}$ then prove that: x – y = $\frac{\pi^c}{3}$.

12. (a) If tan(x + y) = 23 and tan y = 7, what is the value of tan x.

(b) If cos (A – B) = $\frac{84}{85}$ and sec A = $\frac{17}{8}$, find the value of sin B.

(c) Evaluate the value of tan A if cot B = 2m + 1 and A + B = $\frac{\pi^c}{4}$.

13. If A + B = 45°, prove that:

(a) (1 + tanA) (1 – tanB) = 2 (b) (cot A – 1) (cot B – 1) = 2

14. If α – β = $\frac{\pi^c}{4}$, prove that:

(a) (1 – tanα) (1 + tanβ) = 2 (b) (1 – cot α) (1 + cot β) = 2

15. Prove that:

(a) $\frac{\sin(B-C)}{\cos B.\cos C} + \frac{\sin(C-A)}{\cos C.\cos A} + \frac{\sin(A-B)}{\cos A.\cos B} = 0$

(c) sinA.sin(B – C) + sinB.sin(C – A) + sinC.sin(A – B) = 0

16. Prove that the followings:

(a) $\frac{\sin 75^o - \cos 75^o}{\cos 15^o + \cos 75^o} = 1$ (c) $\frac{\sin(45^o + A)}{\cos(45^o - A)} = 1$

(b) $\frac{\tan 75^o + \cot 75^o}{\tan 15^o + \cot 15^o} = 1$ (d) $\frac{\cot(45^o - A)}{\tan(45^o + A)} = 1$

17. Show that the followings:

(a) tan 40° + tan 60° + tan 80° = tan 40°.tan 60°.tan 80°

(b) tan 15°.tan 25° + tan 25°.tan 50° + tan 50°.tan 15° = 1

(c) tan 9A – tan 5A – tan 4A = tan 9A.tan 5A.tan 4A

(d) cot 8A.cot 4A – cot 12A.cot 4A – cot 12A.cot 8A = 1

18. Prove that:

(a) $\frac{\sin(A+B).\sin(A-B)}{\cos^2 A.\cos^2 B} = \tan^2 A - \tan^2 B$

(b) $\frac{\sin(A+B)}{\sin(A-B)} = \frac{\tan A + \tan B}{\tan A - \tan B}$

(c) $\sec(\alpha + \beta) = \frac{\sec\alpha.\sec\beta}{1 - \tan\alpha.\tan\beta}$

(d) $\frac{\sin(A+B) + \sin(A-B)}{\cos(A+B) + \cos(A-B)} = \tan A$

(e) $\tan(A+B) = \frac{\sin^2 A - \sin^2 B}{\sin A.\cos A - \sin B.\cos B}$

19. Prove that:

(b) $\cos(A+B).\cos(A-B) = \cos^2 B - \sin^2 A = \cos^2 A - \sin^2 B$

(a) $\tan(45^o + A) = \frac{1 + \tan A}{1 - \tan A} = \frac{\cos A + \sin A}{\cos A - \sin A}$

(b) $\cot(45^o - A) = \frac{\cot A + 1}{\cot A - 1} = \frac{\cos A + \sin A}{\cos A - \sin A}$

(e) $\tan(A+B) + \tan(A-B) = \frac{\sin 2A}{\cos^2 A - \cos^2 B}$

(f) $\tan(A+B).\tan(A-B) = \frac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}$

20. Prove that:

(a) $\sin(A+B+C) = \cos A.\cos B.\cos C(\tan A + \tan B + \tan C - \tan A.\tan B.\tan C)$

(b) $\cos(A+B+C) = \cos A.\cos B.\cos C(1 - \tan B.\tan C - \tan C.\tan A - \tan A.\tan B)$

(c) $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A.\tan B.\tan C)}{1 - \tan B.\tan C - \tan C.\tan A - \tan A.\tan B}$

(d) $\cot(A+B+C) = \frac{\cot B.\cot C + \cot C.\cot A + \cot A.\cot B - 1}{\cot A.\cot B.\cot C - \cot A - \cot B - \cot C}$.

21. (a) If tanA:tanB = k:1, prove that: sin(A + B):sin(A – B) = (k + 1):(k – 1).

(b) If an angle θ is divided into two parts α and β such that tanα:tanβ :: x:y, verify that: sin θ:sin(α – β) = (x + y):(x – y).

Unit 6

Vectors

6.0 Review

Observe the following statements:

1. The sum of area of two fields is 40 square meter.
2. An aeroplane is flying 20 miles per second towards west.
3. A truck is travelling with 50 km/hr.
4. A car is moving with 35 km/hr starting from Kathmandu bus park. Where will it reach after 1 hour? Estimate.
5. A car is travelling with 35 km/hr starting from Kathmandu Bus park towards Pokhara. Where will it reach after 1 hour? Estimate

Discuss in small groups of students on above conditions based on following questions.

- What is the distance in each case?
- What is the direction of each in each case?
- Which condition contains the distance only?
- Which condition contains the distance and direction both?

Prepare a short report within bench group and present to classroom.

In above statements (2) and (5) both the distance (magnitude) and direction are given. Hence in our daily life every quantity can be measured with the help of number. For some quantities the number is not sufficient for measurement.

6.1 Introduction

The above statements (1) , (3) and (4) have only distance but have not directions. So they are scalar quantities. In statement (2) the magnitude and the directions of aeroplane is given and in statement (5) the magnitude and the direction of care is given. So we called these as vector quantity.

Vector Quantity

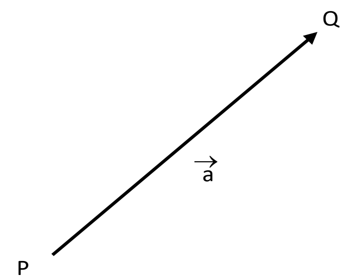
A physical quantity which has both magnitude and direction is called Vector quantity or simply a vector. For example, velocity, acceleration weight, etc.

Scalar Quantity

A physical quantity which has only magnitude is called scalar quantity or simply scalar. For example, volume, length, mass, etc.

Notation of vector

Vectors as displacement are denoted by directed line segment. So, if P and Q be two end points of a vector then the vector from point P to Q is denoted by $\overrightarrow{PQ}$ (two letters with arrow over it) or simply $\vec{a}$ (a single letter a with arrow over it) or **a** (bold faced letters). For example, the vector from point P to Q is denoted by $\overrightarrow{PQ}$ or $\vec{a}$. The magnitude of vector $\overrightarrow{PQ}$ that its length is denoted by $|\overrightarrow{PQ}| = |\vec{a}|$.

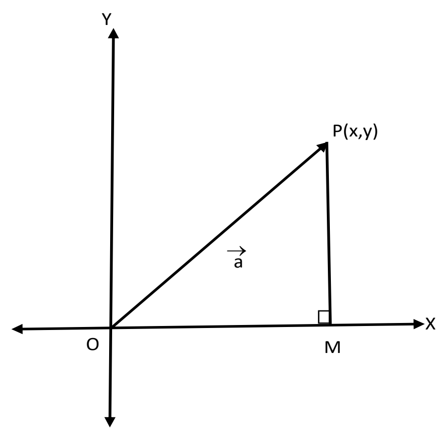


Vector in cartesian plane

Let P(x,y) be any point on the cartesian plane. Join origin O and point P. Draw perpendicular PM on OX (PM⊥OX), so that OM = x, PM = y. The displacement of O to P is same as the displacement from O to M and displacement from M to P.

i.e. Horizontal displacement OM with OM = x and vertical displacement with MP = y, together gives the displacement of OP.

We can write $\overrightarrow{OP}$ as an order pair (x,y) or $\begin{pmatrix} x \\ y \end{pmatrix}$, i.e, $\overrightarrow{OP}$ = (x, y) or $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ is called vector in terms of components. This is also called position vector of point P with respect to fixed point O.



Vector having initial point not at origin

Let P (x_1 ,y_1) and Q (x_2, y_2) be the coordinates of two end point of vector PQ. Draw PM ⊥ OX, QN⊥ OX and PR ⊥ QN.

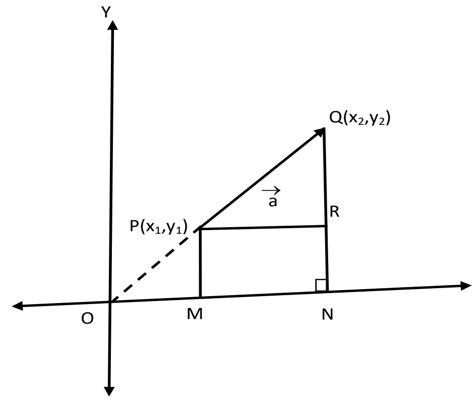
The x-component of PQ is PR = $x_2 - x_1$.

The y-component of PQ is QR = $y_2 - y_1$.

So, the vector joining P (x_1,y_1) and Q (x_2, y_2) is given by

. $$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

or, $\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1)$



Example 1

What will be the meaning of following vector.

a) $\begin{pmatrix}3\\7\end{pmatrix}$ (b) $\begin{pmatrix}3\\-7\end{pmatrix}$

(c) $\begin{pmatrix}-3\\7\end{pmatrix}$ (d) $\begin{pmatrix}-3\\-7\end{pmatrix}$

Solution:

(a) $\begin{pmatrix}3\\7\end{pmatrix}$ means horizontal displacement 3 unit to the right and vertical displacement 7 units upward.

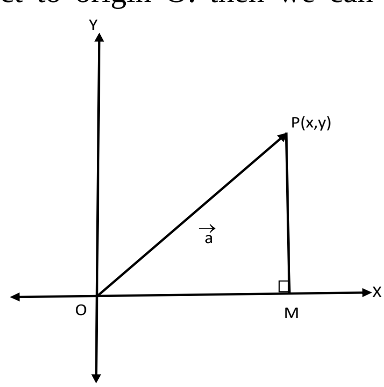
(b) $\begin{pmatrix}3\\-7\end{pmatrix}$ means 3 units horizontal displacement to the right and 7 units vertical displacement downward.

(c) $\begin{pmatrix}-3\\7\end{pmatrix}$ means 3 units horizontal displacement to the left and 7 units vertical displacement upward.

(d) $\begin{pmatrix}-3\\-7\end{pmatrix}$ means 3 units horizontal displacement to the left and 7 units vertical displacement downward.

Magnitude and direction of vector

Let $\overrightarrow{OP}$ be a position vector of point P with respect to origin O. then we can represent it as $\overrightarrow{OP} = \begin{pmatrix}x\\y\end{pmatrix}$



The magnitude of $\overrightarrow{OP}$ is denoted by $|\overrightarrow{OP}|$ and defined by $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$.

or, $|\overrightarrow{OP}| = \sqrt{(x - \text{component})^2 + (y - \text{component})^2}$

Let $\overrightarrow{OP}$ is a vector such that $\angle XOP = \theta$ then θ is called the direction of $\overrightarrow{OP}$.

If PM ⊥ OX, so, in right angle triangle OPM.

$\tan\theta = \frac{PM}{OM} = \frac{y\text{−Component of } \overrightarrow{OP}}{x\text{−Component } \overrightarrow{OP}}$

$\therefore\ \theta = \tan^{-1}\left(\frac{y-component}{x-component}\right)$

If P (x_1, y_1) and Q (x_2, y_2) be two end points of PQ then its direction is given by

$\theta = \tan^{-1}\left(\frac{y_2-y_1}{x_2-x_1}\right)$ and $|\overrightarrow{PQ}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$ is magnitude of $\overrightarrow{PQ}$.

Example 2

If A(2, 5) and B(7,10) be two end points of $\overrightarrow{AB}$. Then

i) Write $\overrightarrow{AB}$ in component form.

ii) Find the magnitude of $\overrightarrow{AB}$.

iii) Find the direction of $\overrightarrow{AB}$.

Solution

Let, A(2,5) B(7,10) be two end points. Then $x_1 = 2$, $x_2 = 7$, $y_1 = 5$, $y_2 = 10$

Now,

(i) The x- component = $x_2 - x_1 = 7 - 2 = 5$

y-component = $y_2 - y_1 = 10 - 5 = 5$

$\therefore\ \overrightarrow{AB} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

(ii) The magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$= \sqrt{(7-2)^2 + (10-5)^2}$

$= \sqrt{25+25}$

$= \sqrt{50} = \sqrt{2 \times 25} = 5\sqrt{2}$

(iii) Direction of $\overrightarrow{AB}$ is given by $\tan\theta = \frac{y_2-y_1}{x_2-x_1}$

or $\tan\theta = \frac{5}{5} = 1$

$\tan\theta = \tan\frac{\pi^c}{4}$

$\therefore \theta = \frac{\pi^c}{4}$

Types of Vectors

(i) Column Vector: Let P(x, y) be a point then the vector $\overrightarrow{OP}$ with O (0,0) and P (x,y) can be expressed as $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ is called column vector.

(ii) Row vector: The vector $\overrightarrow{OP} = (x, y)$ is called row vector.

(iii) Unit vector: Any vector having magnitude one is called a unit vector. i.e. If $|\overrightarrow{OP}| = 1$ then $\overrightarrow{OP}$ is called a unit vector. For example, (1, 0), (0, 1) etc. are unit vectors. If $\vec{a} = (x, y)$ be a vector then the unit vector along $\vec{a}$ is denoted by $\hat{a}$ (a cap) and given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

Example

If $\vec{a} = (3, -4)$, find unit vector along $\overrightarrow{a}$.

Solution: We have, $\vec{a} = (3, -4)$

$$\therefore |\vec{a}| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{(3, \ -4)}{5} = \left(\frac{3}{5}, \frac{-4}{5}\right)$$

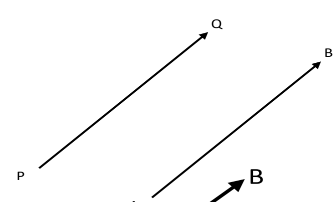
$$\hat{a} = \left(\frac{3}{5}, \frac{-4}{5}\right)$$

(iv) Null or zero vectors:

A vector whose magnitude is zero is called null vector. e.g. (0, 0) is a zero or null vector. If the starting and ending point is same then it is called zero vector.

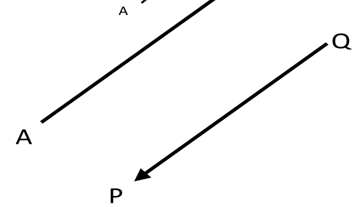
For example, let A = (2, 3) then $\overrightarrow{AA} = \begin{pmatrix} 2-2 \\ 3-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is called the zero (null) vector. It is denoted by O.

(v) Like vectors: If two vectors have the same direction, then these two vectors are called like vectors. For example $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are like vectors.



(iv) Unlike vectors: If two vectors have the opposite direction, then these two vectors are called unlike vectors.

In given figure $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are unlike vectors.



Note: *If we can express two vectors as* $\overrightarrow{AB} = k\,\overrightarrow{PQ}$, $k \neq 0$ *then two vectors are like if k is positive and two vectors* *are unlike if k is negative.*

If two vectors are like or unlike vector, then they are called parallel vectors. The collinear vectors are also parallel.

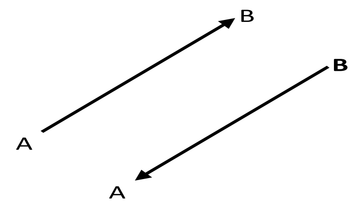
(vii) Equal vector: Two like vector having both magnitude and direction equal (same) are called equal vectors.

or, if $\overrightarrow{AB}$ = k $\overrightarrow{PQ}$*: and k = 1 then they are equal vectors.*

i.e. if $\overrightarrow{a} = (a_1,\ a_2)$ and $\vec{b} = (b_1,\ b_2)$ then $\vec{a} = \vec{b}$ if $a_1 = b_1$ and $a_2 = b_2$

(viii) Negative of a vector: If $\vec{a} = (a_1, a_2)$ be a vector. Then the negative of $\vec{a}$ is denoted by $\overrightarrow{-a}$ and given by $\overrightarrow{-a} = (-a_1, -a_2)$.

note: The magnitude of $\overrightarrow{a}$ *is always equal to the magnitude of* -$\vec{a}$*, but their direction is taken opposite.*



The negative of $\overrightarrow{AB}$ = –$\overrightarrow{AB}$ = $\overrightarrow{BA}$.

Example 4

Draw directed line segment to represent the following vectors.

(a) $\overrightarrow{\mathrm{OP}} = (-5,4)$ (b) $\overrightarrow{\mathrm{OQ}} = (3,4)$

(b) $\overrightarrow{\mathrm{OR}} = (-2,-3)$ (d) $\overrightarrow{\mathrm{OS}} = (5,-3)$

Solution:

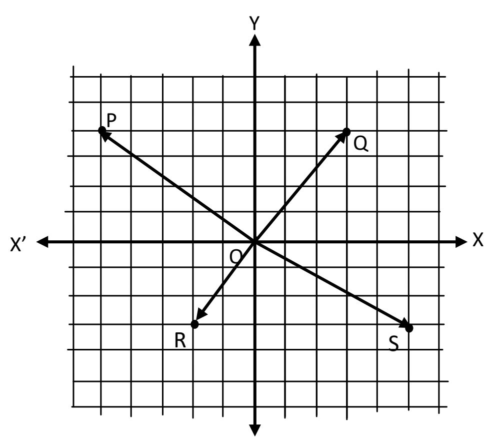
We have the directed line segment are as follows:

:$\overrightarrow{\mathrm{OP}} = (-5,4) = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

$\overrightarrow{\mathrm{OQ}} = (3,4) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\overrightarrow{\mathrm{OR}} = (-2,-3) = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$\overrightarrow{\mathrm{OS}} = (5,-3) = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$



Example 5

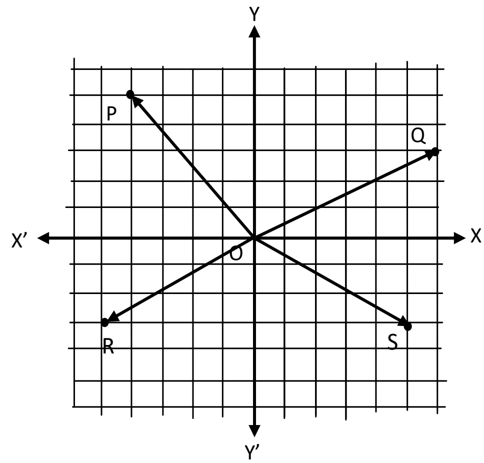
From given directed line segment find x-component and y-component and then position vector of each point.

(a) P

(b) Q

(c) R

(d) S



Solution:

For point P,

x-component = horizontal displacement = –4

y-component = vertical displacement = 5

Therefore $\overrightarrow{OP} = (-4, 5) = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

For point Q

x-component = horizontal component = 6

y-component = vertical component = 3

$\therefore \overrightarrow{OQ} = (6,\ 3) = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

For point R

x-component = horizontal component = –5

y-component = vertical component = –3

$\therefore \overrightarrow{OR} = (-5, -3) = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$

For point S

x-component = 5, y-component = –3

$\therefore \overrightarrow{OS} = (5, -3) = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

Example 6

Find magnitude, direction, units vector and negative vector of the following vectors.

(a) (4, 4) **(b)** $\left(-2\sqrt{3}, 2\right)$

Solution: Here,

(a) We have the given vector is (4, 4)

i.e. $\vec{a} = (4, 4)$

Then magnitude of $\vec{a} = |\vec{a}| = \sqrt{4^2 + 4^2} = \sqrt{16 + 16} = \sqrt{2 \times 16} = 4\sqrt{2}$

Unit vector along $\vec{a} = \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \left(\frac{4}{4\sqrt{2}}, \frac{4}{4\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

The direction of $\vec{a}$ is $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{1}\left(\frac{4}{4}\right) = \tan^{-1}(1) = 45° = \frac{\pi^c}{4}$

The negative of $\vec{a}$ is $-(\vec{a}) = (-4, -4)$

(b) Let $\vec{b} = (-2\sqrt{3}, 2)$ then

(c) $|\vec{b}| = \sqrt{\left(-2\sqrt{3}\right)^2 + 2^2} = \sqrt{4 \times 3 + 4} = \sqrt{16} = 4$

Unit vector along $\vec{b} = \hat{b} = \frac{\vec{b}}{|\vec{b}|} = \left(\frac{-2\sqrt{3}}{4}, \frac{2}{4}\right) = \left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$

Direction of $\vec{b} = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$= \tan^{-1}\left(\frac{2}{-2\sqrt{3}}\right)$

$= \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \frac{5\pi^c}{6}$.

and negative of $\vec{b} = -(\vec{b}) = -\left(-2\sqrt{3}, 2\right) = \left(2\sqrt{3}, -2\right)$

Example 7

Let A(3, 3), B(6, 0), C(3, –3) and D(*x*, y) be four points in a plane. If $\overrightarrow{AB} = \overrightarrow{CD}$ find the coordinates of point D.

Solution: Here,

$\overrightarrow{AB}$ displaces A(3, 3) to B(6, 0) and $\overrightarrow{CD}$ displace (3, –3) to (x, y)

Then, $\overrightarrow{AB} = (x_2 - x_1,\ y_2 - y_1) = (6 - 3, 0 - 3) = (3, -3) = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

and $\overrightarrow{CD} = (x_2 - x_1, y_2 + y_1) = (x - 3, y + 3) = \begin{pmatrix} x - 3 \\ y + 3 \end{pmatrix}$

Since $\overrightarrow{AB} = \overrightarrow{CD}$

so, $\begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} x-3 \\ y+3 \end{pmatrix}$

or, $x - 3 = 3$ and $y + 3 = -3$

$x = 6$ and $y = -6$

$D(x, y) = (6, -6)$

Exercise 6.1

1. **Draw directed line segment taking origin as a initial point and the following points as a terminal point in grid paper.**

   (a) A(4, 7) (b) B(8, –3) (c) C(6, -5)

   (d) D(8, 3) (e) L(–5, 8) (d) M(-4, -7)

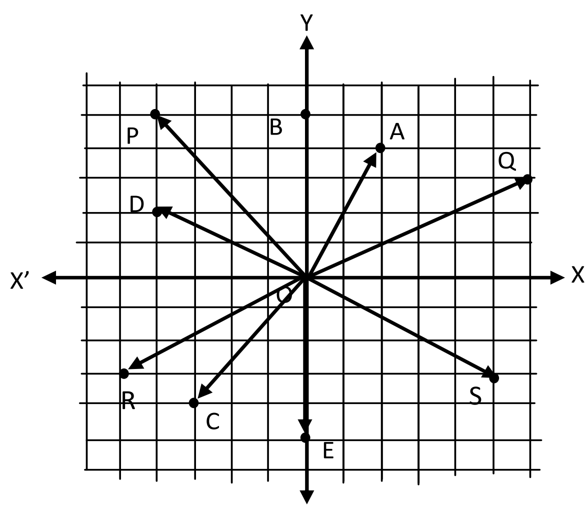
2. **Find the vector represented by the directed line segments joining the following points.**

   (a) A(5, 3) and B(7, 4) (b) P(8, –7) and A(5, 4)

   (c) M(–6, –8) and O(0, 0) (d) B(7, –1) and C(9, 2)

   (e) K(6, 2) and L(5, –2) (f) E(2, 1) and F(1, 2)

3. **Find x-component and y-component of each directed line segments given in adjoining figure.**



4. **Find magnitude, direction and negative vector of each of the vector in Q.2.**

5. The position vectors of A and B are given below. Find column vector of $\overrightarrow{AB}$ in each case and find unit vector along $\overrightarrow{AB}$.

(a) $\overrightarrow{OA} = (-3, 4), \overrightarrow{OB} = (6, 3)$ (b) $\overrightarrow{OA} = (3, 5), \overrightarrow{OB} = (2, -5)$

(b) $\overrightarrow{OA} = (-6, -2)$ and $\overrightarrow{OB} = (9, 3)$ (d) $\overrightarrow{OA} = (2, 2), \overrightarrow{OB} = (-5, 6)$

6. (a) If $\overrightarrow{PQ}$ displaces P (3, 5) to Q (2, 5) and MN displaces M (1, 3) to N (3, 0) prove that $\overrightarrow{PQ} = \overrightarrow{MN}$.

(b) If $\overrightarrow{AB}$ displaces A(3, 4) to B(4, 7) and $\overrightarrow{XY}$ displaces X(6, 3) to Y(5, 0) prove that $\overrightarrow{AB} = -\overrightarrow{XY}$

(c) If P(2, 4), Q(6, 3), R(–3, 5) and S(1, 4) are four points, prove that $\overrightarrow{PQ} = \overrightarrow{RS}$.

7. (a) If A(6, 4), B(3, –5), C(2, –2) and D(x, y) be four points such that $\overrightarrow{AB} = \overrightarrow{CD}$ find coordinates of D.

(b) If AP displaces A(9, 8) to P(5, 4) and BQ displaces B (8, –1) to the point Q such that $\overrightarrow{AP} = \overrightarrow{BQ}$. Find coordinates of Q.

(c) Find value of *x* and *y* such that $\vec{a} = \begin{pmatrix} 3x+2 \\ 6 \end{pmatrix}, \vec{b} = \begin{pmatrix} 8 \\ 2y+2 \end{pmatrix}$ and $\vec{a} = \vec{b}$.

(d) If point A (3, 1) is displaces to B (1, 4) by $\overrightarrow{AB}$, $\overrightarrow{PQ}$ displaces P(2,2) to Q (*x*, *y*) and $\overrightarrow{AB} = \overrightarrow{PQ}$ find the value of (*x*, *y*).

6.2 Operation of vectors

Like as number and algebraic expressions we can operate two vectors. We can add, subtract and multiply two vectors. Such types of operations are called vector operation. In this topic we will discuss about the following:

(i) Multiplication of vector by a scalar

(ii) Addition of two vectors.

(iii) Subtraction of two vectors.

(h) Multiplication of a vector by a scalar

Let $\overrightarrow{PQ} = \vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ and k be a scalar quantity than the multiplication of $\overrightarrow{PQ} = \vec{a}$ by k is denoted by $k\,\vec{a}$ and defined by

$k\,\vec{a} = k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$

Since the direction of a is $\tan^{-1}\left(\frac{y}{x}\right)$ and direction is of $k\vec{a}$ is $\tan^{-1}\left(\frac{ky}{kx}\right) = \tan^{-1}\left(\frac{y}{x}\right)$.

So they have same direction. Hence $k\vec{a}$ is parallel to $\vec{a}$ i.e $k\vec{a} \,||\, \vec{a}$ and

$|\vec{a}| = \sqrt{x^2 + y^2}$

$|k\vec{a}| - \sqrt{(kx)^2 + (ky)^2} = \sqrt{k^2x^2 + k^2y^2}$

$= k\sqrt{x^2 + y^2 = k\,|\vec{a}|}$

The magnitude of $k\vec{a}$ is k time the magnitude of $\vec{a}$.

Example 1

If $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$, find 3 $\overrightarrow{PQ}$ and show that $|3\overrightarrow{PQ}| = 3|\overrightarrow{PQ}|$.

Solution: We have,

Since, $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$ then $3\overrightarrow{PQ} = 3\begin{pmatrix} -4 \\ 7 \end{pmatrix} = \begin{pmatrix} -12 \\ 21 \end{pmatrix}$

Now $|\overrightarrow{PQ}| = \sqrt{(-4)^2 + 7^2} = \sqrt{16 + 49} = \sqrt{65}$

$|3\,\overrightarrow{PQ}| = \sqrt{(-12)^2 + (21)^2}$

$= \sqrt{144 + 441}$

$= \sqrt{585}$

$= \sqrt{9 \times 65}$

$= 3\sqrt{65}$

$= 3|\overrightarrow{PQ}|$

$\therefore |3\,\overrightarrow{PQ}| = 3|\overrightarrow{PQ}|$

(ii) Addition of two vectors

If $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ be two column vectors then their sum is denoted by $\vec{a} + \vec{b}$ and defined by $\vec{a} + \vec{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$

Hence, two vectors are added by adding their corresponding components.

Example 2

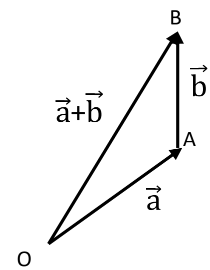
If $\overrightarrow{PQ} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$ find $\overrightarrow{PQ} + \overrightarrow{AB}$

Solution:

We have $\overrightarrow{PQ} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

$\therefore \overrightarrow{PQ} + \overrightarrow{AB} = \begin{pmatrix} -3 \\ 8 \end{pmatrix} + \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} -3+7 \\ 8+3 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}$

Triangle law of vector addition



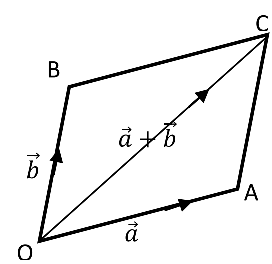
Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$ such that the end point of $\vec{a}$ is starting point of $\vec{b}$. Then the vector represented by the directed line segment joining the starting point of $\vec{a}$ and terminal point $\vec{b}$ is called the sum of $\vec{a}$ and $\vec{b}$. This law is called the triangle law of vector addition.

$\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$

Then, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

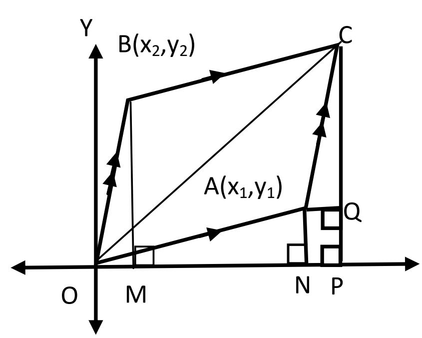
$\overrightarrow{OB} = \vec{a} + \vec{b}$

Parallelogram law of vector addition



If two vectors represent the adjacent sides of a parallelogram, then their sum is represented by the diagonal of the parallelogram leading from the starting point of vectors. It is self-evident by triangle law of vector addition.

Geometric interpretation of Parallelogram law of vector addition



Let $\overrightarrow{OA} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be two vectors represented as the adjacent sides of parallelogram OACB.

Draw BM $\perp$ OX, AN$\perp$ OX, CP $\perp$ OX and AQ $\perp$ CP. Then from figure,

OM = x_2, BM = y_2 ON = x_1, AN = QP = y_1

In right angled triangles OMB and CAQ

OB = AC, ∠BMO = ∠CQA, ∠BOM = ∠CAQ

$\therefore \Delta$OMB $\cong \Delta$CAQ

BM = CQ = y_2, NP = AQ = OM = x_2

Now, OP = ON + NP = $x_1 + x_2$

CP = PQ + CQ = $y_1 + y_2$

The coordinates of C is $(x_1 + x_2, y_1 + y_2)$ and

$$\overrightarrow{\mathrm{OC}} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

Example 3

If $\vec{\mathrm{a}} = (2, 3)$ and $\vec{\mathrm{b}} = (1, -2)$, then find $\vec{\mathrm{a}} + \vec{\mathrm{b}}$.

> Column vectors or row vectors are added by adding their corresponding components.

Solution:

We have $\vec{\mathrm{a}}$ = (2, 3), $\vec{\mathrm{b}}$ = (1, –2)

Then $\vec{\mathrm{a}} + \vec{\mathrm{b}}$ = (2, 3) + (1, –2)

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 + 1 \\ 3 - 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

(iii) Difference of two vectors

Let $\overrightarrow{\mathrm{OA}} = \vec{\mathrm{a}}$ and $\overrightarrow{\mathrm{OB}} = \vec{\mathrm{b}}$ be two vectors. Then their difference is denoted by $\overrightarrow{\mathrm{OA}} - \overrightarrow{\mathrm{OB}} = \vec{\mathrm{a}} - \vec{\mathrm{b}}$ and defined by $\vec{\mathrm{a}} - \vec{\mathrm{b}} = \vec{\mathrm{a}} + (-\vec{\mathrm{b}})$

If $\overrightarrow{\mathrm{OA}} = \vec{\mathrm{a}} = (a_1, a_2)$ and

$\overrightarrow{\mathrm{OB}} = \vec{\mathrm{b}} = (b_1, b_2)$ then

$-\overrightarrow{\mathrm{OB}} = -\vec{\mathrm{b}} = (-b_1, -b_2)$

$$\vec{\mathrm{a}} - \vec{\mathrm{b}} = \vec{\mathrm{a}} + (-\vec{\mathrm{b}}) = (a_1, a_2) + (-b_1, -b_2) = (a_1 - b_1, a_2 - b_2)$$

$$\vec{\mathrm{a}} - \vec{\mathrm{b}} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$$

Example 4

If $\vec{\mathrm{a}} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$ and $\vec{\mathrm{b}} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ find $\vec{\mathrm{a}} - \vec{\mathrm{b}}$.

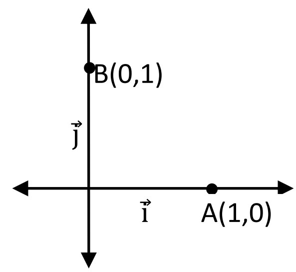
Solution: Here,

We have $\vec{a} = \begin{pmatrix}7\\5\end{pmatrix}, \vec{b} = \begin{pmatrix}4\\2\end{pmatrix} \Rightarrow -\vec{b} = \begin{pmatrix}-4\\-2\end{pmatrix}$

Now $\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = \begin{pmatrix}7\\5\end{pmatrix} + \begin{pmatrix}-4\\-2\end{pmatrix} = \begin{pmatrix}7-4\\5-2\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$

Alternatively $\vec{a} - \vec{b} = \begin{pmatrix}7\\5\end{pmatrix} - \begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}7-4\\5-2\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$

Unit Vector $\vec{i}$ and $\vec{j}$.



Let O be the origin OX is positive X-axis and OY is positive Y-axis then the unit vectors along OX and OY respectively are denoted by $\vec{i}$ and $\vec{j}$.

For $\vec{i}$, x-component = 1 and y-component = 0 so $\vec{i}$ = (1,0)
For $\vec{j}$, x-component = 0 and y component = 1 so $\vec{j}$ = (0,1)

for any vector, $\vec{r}$ = (a, b), we can express (a, b) = $a\vec{i} + b\vec{j}$ and conversely.

For, $\vec{r}$ = (a, b) we have $\vec{i} = (1, 0), \vec{j} = (0, 1)$ then

$a\vec{i} + b\vec{j}$ = a (1, 0) + b (0, 1)

= (a, 0) + (0, b) = (a + 0, 0 + b) = (a, b)

Conversely, (a, b) = (a + 0, 0 + b) = (a, 0) + (0, b)

= a(1, 0) + b(0, 1)

= $a\vec{i} + b\vec{j}$

Example 5

Express $\vec{a}$ = (–3, 7) in terms of $\vec{i}$ and $\vec{j}$.

Solution: Here,

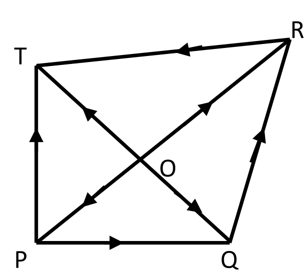
$\vec{a} = (-3, 7)$ i.e. a = –3, b = 7

We have,

(a, b) = $a\vec{i} + b\vec{j}$

(–3, 7) = $-3\vec{i} + 7\vec{j}$

Example 6



In adjoining figure $\overrightarrow{PQ} = \vec{a}, \overrightarrow{QR} = \vec{b}$ if O is the mid-point of $\overrightarrow{PR}$, find $\overrightarrow{PR}, \overrightarrow{PO}, \overrightarrow{RO}$ and $\overrightarrow{QO}$.

Solution

Since PQR is a triangle so by triangle law of vector addition

$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \left(\vec{a} + \vec{b}\right)$

$\overrightarrow{PO} = \frac{1}{2}\overrightarrow{PR} = \frac{1}{2}\left(\vec{a} + \vec{b}\right)$ ($\because$ O is midpoint of PR)

$$\overrightarrow{PO} = \overrightarrow{OR} \Rightarrow \overrightarrow{RO} = -\overrightarrow{PO} = -\frac{1}{2}\left(\vec{a} + \vec{b}\right)$$

Now, $\overrightarrow{QO} = \overrightarrow{QP} + \overrightarrow{PO} = -\overrightarrow{PQ} + \overrightarrow{PO} = -\vec{a} + \frac{1}{2}\left(\vec{a} + \vec{b}\right)$

$$= \frac{-2\vec{a} + \vec{a} + \vec{b}}{2} = \frac{\vec{b} - \vec{a}}{2} = \frac{1}{2}\left(\vec{b} - a\right)$$

Example 7

Prove that (4, 3), (6, 4), (5, 6) represent the vertices of isosceles triangle.

Solution: Here, we have,

$\overrightarrow{OA} = (4, 3), \overrightarrow{OB} = (6, 4), \overrightarrow{OC} = (5, 6)$

Now,

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (6, 4) - (4, 3) = (6 - 4, 4 - 3) = (2, 1)$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (5, 6) - (6, 4) = (5, -6, 6 - 4) = (-1, 2)$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = (4, 5) - (5, 6) = (4 - 5, 3 - 6) = (-1, -3)$

Now, $\left|\overrightarrow{AB}\right| = \sqrt{2^2 + 1^2} = \sqrt{4 + 1} = \sqrt{5}$

$\left|\overrightarrow{BC}\right| = \sqrt{(-1)^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}$

$\left|\overrightarrow{CA}\right| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{1 + 9} = \sqrt{10}$

Since $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right| = \sqrt{5}$ units, so ΔABC is an isosceles triangle.

Hence A, B, C are vertices of an isosceles triangle.

Exercise 6 .2

1. If $\vec{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -5 \\ 6 \end{pmatrix}$ find:

(a) $\overrightarrow{3a}$ (b) $-2\vec{a}$ (c) $4\vec{b}$ (d) $-5\vec{b}$

(e) $\left|\overrightarrow{5a}\right|$ (f) $\left|2\vec{b}\right|$ (g) $2|\vec{a}| + \left|\vec{b}\right|$

2. **Find sum of the following vectors:**

(a) $\vec{a} = (3, 5)$ and $\vec{b} = (3, 4)$ (b) $\vec{a} = (-3, -4)$ and $\vec{b} = (-3, 2)$

(c) $\vec{a} = (5, -7)$ and $\vec{b} = (3, 2)$ (d) $\vec{a} = (4, 5)$, $\vec{b} = (-4, -5)$

(e) $\vec{p} = (-5, 5)$ and $\vec{q} = (3 - 2)$ (f) $\vec{c} = (3, 8)$ and $\vec{b} = (3, -2)$

3. **If $\vec{a} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ then find:**

(a) $2\vec{a} + 3\vec{b}$ (b) $3\vec{a} - 2\vec{b}$ (c) $|2\vec{a} + 2\vec{b}|$ (d) $\vec{a} + \vec{b}$

(e) $\vec{a} - \vec{b}$ (f) $|\vec{a} - \vec{b}|$ (g) $|\vec{a} + \vec{b}|$

4. **If $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$, $\vec{c} = (c_1, c_2)$, then verify the following:**

(a) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (b) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

(c) $\vec{c} + (-\vec{c}) = 0$ (d) $2(\vec{a} + \vec{c}) = 2\vec{a} + 2\vec{c}$

5. (a) Express (–4, 5) in terms of $\vec{i}$ and $\vec{j}$ and conversely.

(b) Express $3\vec{i} + 4\vec{j}$ in terms of (x,y) and conversely.

(c) If the position vector of P and Q are $4\vec{i} + 6\vec{j}$ and $5\vec{i} + 3\vec{j}$ respectively find $\overrightarrow{PQ}$. its magnitude and unit vector along PQ.

(d) If the coordinates of P and Q are (4, 3) and (–2,4) then find $\overrightarrow{PQ}$, $|\overrightarrow{PQ}|$, $\widehat{PQ}$.

6. Show that the following sets of points are vertices of a right angled triangle.

(a) (2, 1), (3, 0) and (1, 0) (b) (6, 4), (6, 7) and (2, 4)

(c) (–2, 5), (3, -4) and 7, 10)

7. Show the following sets of points are vertices of an isosceles triangle.

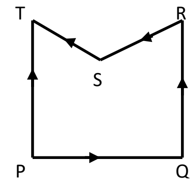
(a) (2, 3) (2, 0) and (–1, 0) (b) (0, 4), (3, –3) and (–3, 3)

(c) (5, 5), (5, 0) and (0, 0) (d) (2, 0), (–1, 0) and (–3, 3)

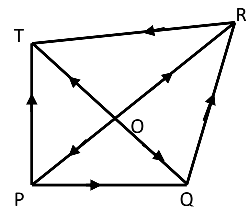
8. (a) If PQRST be a regular pentagon then show that $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST} = \overrightarrow{PT}$

(b) If ABCDEF be a hexagon. Prove that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = 0$

(c) In adjoining figure, prove that $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RS} + \overrightarrow{ST} = \overrightarrow{PT}$



(d) In adjoining figure ABCD is a quadrilateral with diagonal AC and BD intersecting at O. If $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$ and $\overrightarrow{OD} = \vec{d}$. Find $\overrightarrow{AB}$, $\overrightarrow{BC}, \overrightarrow{CD}$ and $\overrightarrow{DA}$ in terms of a, b, c and d and prove that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$.



Unit 7

Transformation

7.0 Review

What is transformation? How many types of transformation are there? What are they?

What is reflection? Tell some examples of reflection that are used in our daily life activities.

Which transformations have similar or congruent images with their objects?

What are the coordinates of the images of P(x, y) under reflection on the lines x = 0, y = 0, x = y and y = – x?

What are the coordinates of the images of P(x, y) when rotated about the origin through the angles 90°, 180° and 270° separately?

7.1 Transformation and its types

(i) When you look mirror, where does the image form?

(ii) If you go to school, it takes 15 minutes from your home. In this time period, at what angle does the minute hand move?

(iii) When you drag a book on the bench from one edge to another edge, what do you see?

(iv) When any object is place in the front of a round mirror, where does its image form?

A change in position or size of an object is called its transformation. The position or size of the object after transformation is called its image.

There are mainly two types of transformation. They are:

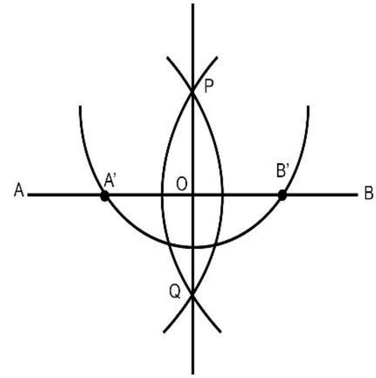
(i) Isometric Transformation (ii) Non-isometric Transformation

Isometric Transformation: A transformation, in which the object and its image are congruent, is called isometric transformation. Furthermore, the isometric transformations are classify into three types: reflection, rotation and translation.

Non-isometric Transformation: A transformation, in which the object and its image are similar, is called non-isometric transformation.

7.1 Reflection

A reflection is a transformation in which a geometric figure is reflected across a line, called the reflecting axis and the perpendicular distance from the axis to the object and its image are the same. For example, in the adjoining figure, Q is the image of a point P in the reflecting axis AB and PO = QO when drawing PQ⊥AB by using compass.



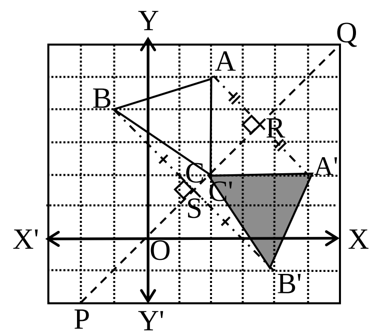
Properties of reflection

The properties of reflection are:

(i) Points on mirror line are invariant. In the figure C is invariant point.

(ii) A reflection preserves lengths of segments. In the adjoining reflection of ΔABC on the line PQ, AR = A'R and BS = B'S.

(iii) Object and image are reverse to each other. In the figure, ΔABC and ΔA'B'C' are reverse to each other.



(iv) Object and image under the reflection are congruent. In the figure, ΔABC ≅ΔA'B'C'.

(v) Lines perpendicular to the mirror lines are invariant but points on them are not invariant. In the figure, AA'⊥PQ, BB'⊥PQ and BB'⊥PQ.

Example 1

Reflect ΔABC in the given reflecting line *l*:

Solution: Here,

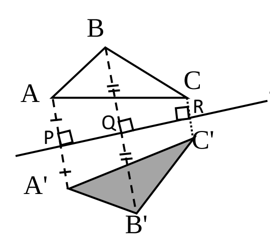
Steps of reflection:

i) From A, draw AP ⊥ *l* and taking AP = A'P , produce to A'

ii) From B, draw BQ ⊥ *l* and produce to B', taking BQ = B'Q

iii) From C, draw CR ⊥ *l* and produce to C' , taking CR = C'R

iv) Join A' B' C'

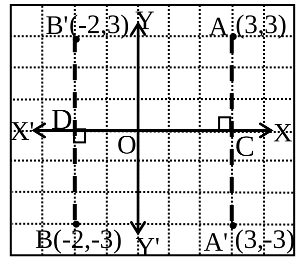


Hence Δ A' B' C' is the image of ΔABC after reflecting line *l*.

Reflection in cartesian plane

(i) Reflection in the X-axis or the line y = 0

Discuss the reflection in the X-axis or the line y = 0 in the given adjoining graph.



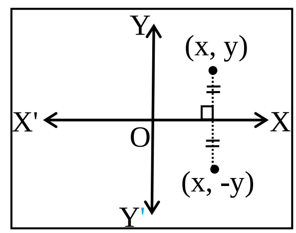
In the graph, the coordinates of A' is (3, –3).

i.e., A(3, 3) $\xrightarrow{Re;\, x-axis}$ A'(3, –3).

Similarly, the image of B(–2, –3) under the reflection on x-axis is B'(–2, 3). i.e.,

B(-2, -3) $\xrightarrow{Re;\, x-axis}$ B'(–2, 3).

Hence, the image of any point under the reflection in X-axis is obtained by changing the sign of the y-coordinate of the given point.

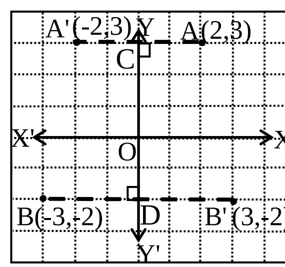


∴ The image of the point (x, y) under the reflection in X-axis is the point (x, –y).

i.e., **(x, y)** $\xrightarrow{\boldsymbol{Re;\, x-axis}}$ **(x, – y)**

(ii) Reflection in the Y-axis or the line x = 0

Discuss the reflection in the y-axis or the line x = 0 in the given adjoining graph.

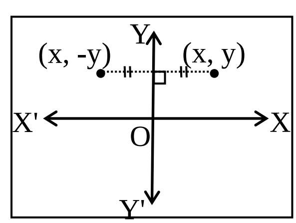


In the graph, the coordinate of A' is (2, 3).

i.e., A(2, 3) $\xrightarrow{\boldsymbol{Re;\, y-axis}}$ A'(–2, 3).

Similarly, the image of B(-3, -2) under the reflection in y-axis is B'(3, –2). i.e., B(–3, –2) $\xrightarrow{\boldsymbol{Re;\, y-axis}}$ B'(3, –2)

Hence, the image of any point under the reflection in Y-axis is obtained by changing the sign of the x-coordinate of the given point.

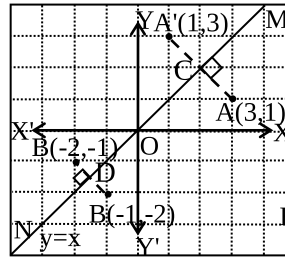


∴ The image of the point (x, y) under the reflection in Y-axis is the point (–x, y).

i.e., **(x, y)** $\xrightarrow{\boldsymbol{Re;\, y-axis}}$ **(–x, y)**

(iii) Reflection in the line y = x or the line y – x = 0

Discuss the reflection in the line y = x or the line y – x = 0 in the given adjoining graph.



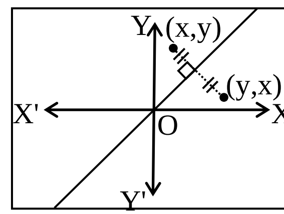
In the graph, the coordinates of A' is (1, 5).

i.e., A(5, 1) $\xrightarrow{Re;\, y = x}$ A'(1,5)

Similarly, the image of B(–3, –1) under reflection in the line y = x is B'(–1, –3).

i.e., B(–3, –1) $\xrightarrow{Re;\, y = x}$ (–1, –3).

Hence, the image of any point under the reflection in the line y = x is obtained by interchanging the x-coordinate and y-coordinate of the given point.

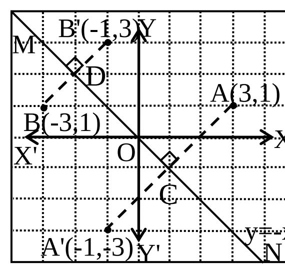


∴ The image of the point (x, y) under the reflection in the line y = x or x – y = 0 is the point (y, x).

i.e., **(x, y)** $\xrightarrow{Re;\, y = x}$ **(y, x)**

(iv) Reflection in the line y = – x or the line y + x = 0

Discuss the reflection in the line y = – x or the line y + x = 0 in the given adjoining graph.



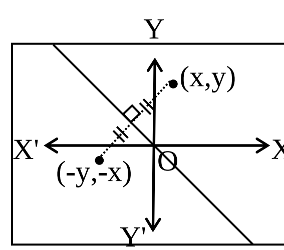
In the graph, the coordinates of A' is (–1, –3).

i.e., A(3, 1) $\xrightarrow{Re;\, y = -x}$ A'(–1, –3).

Similarly, the coordinates of the point B(–3, 1) under the reflection in the line y = – x is B'(–1, 3).

i.e., B(–3, 1) $\xrightarrow{Re;\, y = -x}$ B'(–1, 3).

Hence, the image of any point under the reflection in the line y = –x is obtained by interchanging the x-coordinate and y-coordinate with opposite signs of the given point.

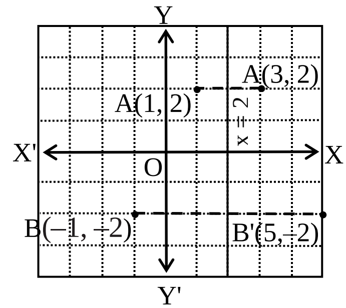


∴ The image of the point (x, y) under the reflection in the line y = – x or x + y = 0 is the point (–y, –x).

i.e., **(x, y)** $\xrightarrow{Re;\, y = -x}$ **(–y, –x)**

(v) Reflection in the line x = a or x – a = 0 (Parallel to y-axis)

Discuss the reflection in the line x = a or the line x – a = 0 in the given adjoining graph.

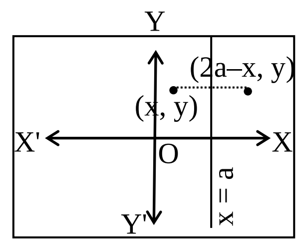


In the graph, the coordinates of the point A(1, 2) under the reflection in the line x = 2 is A'(3, 2).

i.e., A(1, 3) $\xrightarrow{\boldsymbol{Re};\, \boldsymbol{x = 2}}$ A'(3, 2) = A'(2 × 2 –1, 2).

Similarly, the coordinates of the point B(–1, –2) under the reflection in the line x = 2 is B'(5, –2).

i.e., B(–1, –2) $\xrightarrow{\boldsymbol{Re};\, \boldsymbol{x = 2}}$ B'(5, –2) = B'(2 × 2 – (–1), –2).



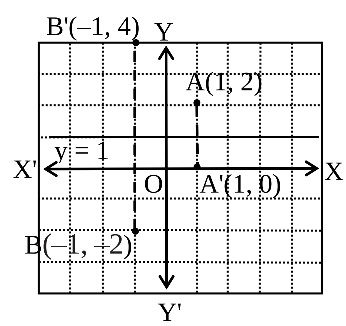
Hence, the image of any point under the reflection in the line x = a is obtained by changing the x-coordinate of the given point by (2a – x).

∴ The image of the point (x, y) under the reflection in the line x = a or x – a = 0 is the point (2a – x, y).

i.e., (x, y) $\xrightarrow{Re;\, x = a}$ (2a – x, y)

(vi) Reflection in the line y = b or y – b = 0 (Parallel to x-axis)

Discuss the reflection in the line y = b or the line y – b = 0 in the given adjoining graph.

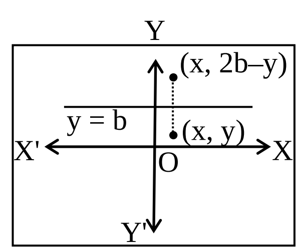


In the graph, the coordinates of the point A(1, 2) under the reflection in the line y = 1 is A'(1, 0).

i.e., A(1, 2) $\xrightarrow{\boldsymbol{Re};\, \boldsymbol{y = 1}}$ A'(1, 0) = A'(1, (2 × 1 –2)).

Similarly, the coordinates of the point B(–1, –2) under the reflection in the line y = 1 is B'(–1, 4).

i.e., B(–1, –2) $\xrightarrow{\boldsymbol{Re};\, \boldsymbol{y = 1}}$ B'(–1, 4) = B'(–1, (2 × 1 –(–2)).



Hence, the image of any point under the reflection in the line y = b is obtained by changing the y-coordinate of the given point by (2b – y).

∴ The image of the point (x, y) under the reflection in the line y = b or y – b = 0 is the point (x, 2b – y).

i.e., $(x, y) \xrightarrow{Re;\, y = b} (x, 2b - y)$

Rules of Reflection in the given reflection axis summarized below:

SN	Object	Axis of Reflection	Image
1.	P(x, y)	$\xrightarrow{\text{x-axis or } y = 0}$	P'(x, -y)
2.	P(x, y)	$\xrightarrow{\text{y-axis or } x = 0}$	P'(-x, y)
3.	P(x, y)	$\xrightarrow{y = x \text{ or } x - y = 0}$	P'(y, x)
4.	P(x, y)	$\xrightarrow{y = -x \text{ or } x + y = 0}$	P'(-y, -x)
5.	P(x, y)	$\xrightarrow{x = a \text{ or } x - a = 0}$	P'(2a – x, y)
6.	P(x, y)	$\xrightarrow{y = b \text{ or } y - b = 0}$	P'(x, 2b – y)

Example 2

Find the coordinates of image of a point (4, –2) under the reflection in x-axis.

Solution: Here,

The given point is (4, –2)

Now, we have

$$(x, y) \xrightarrow{Re,\ x-axis} (x, -y)$$

$$\therefore\ (4, -2) \xrightarrow{\text{x−axis}} (4, 2)$$

Hence, the required coordinates of the image of the given point (4, –2) under the reflection in x-axis is (4, 2).

Example 3

The image of A(3p + 1, 3 – q) under the reflection in the line y = x is A'(q + 1, p + 5). Find the values of p and q.

Solution: Here,

We have, reflecting A(3p + 1, 3 – q) in the line y = x,

$$A(3p + 1, 3 - q) \xrightarrow{Re;\, x=y} A'(3 - q, 3p + 1)$$

Since $A(3p + 1, 3 - q) \xrightarrow{Re;\, x=y} A'(q + 1, p + 5)$,

so (3 – q, 3p + 1) = (q + 1, p + 5)

∴ 3 – q = q + 1 and 3p + 1 = p + 5

or, 3 – 1 = q + q and 3p – p = 5 – 1

or, 2 = 2q and 2p = 4

or, q = 1 and p = 2.

Example 4

Find the reflecting axis in which the point A(–4, –4) reflects into A'(4, – 4).

Solution

Since the axis reflects the point A(– 4, – 4) to A'(4, – 4). So, the midpoint of AA' is $\left(\frac{-4+4}{2}, \frac{-4-4}{2}\right)$ = (0, – 4).

Hence, these points have the same y-component –4 and x-component 0. So, the equation of the axis of reflection is x = 0 or y-axis.

Example 5

Reflect ΔPQR having the vertices P(–2, 3), Q(0, 1) and R(3, 2) in the line x – y = 0 and then write the coordinates of the vertices of the image ΔP'Q'R'. Represent the above reflection on the same graph.

Solution: Here,

Given, the vertices of ΔPQR are P(–2, 3), Q(0, 1) and R(3, 2).

Now, we have

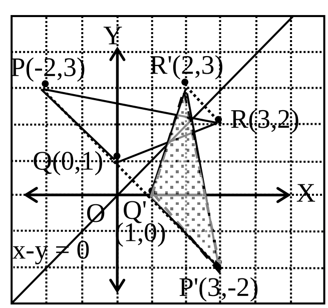
$$(x, y) \xrightarrow{Re;\, x-y=0} (y, x)$$

so, the coordinates of the vertices of the image ΔP'Q'R' are

$$P(-2, 3) \xrightarrow{Re;\, x-y=0} P'(3, -2)$$

$$Q(0, 1) \xrightarrow{Re;\, x-y=0} Q'(1, 0)$$

$$R(3, 2) \xrightarrow{Re;\, x-y=0} R'(2, 3)$$



Hence, the coordinates of the vertices of the image ΔP'Q'R'

are P'(–3, 2), Q'(1, 0) and R'(2, 3).

The reflection of triangle PQR and its image are given in adjoining graph.

Exercise 7.1

1. **Answer of the following questions in single sentence:**

   (a) What is transformation?

   (b) Name the types of transformation.

   (c) What are the types of isometric transformation? Write them.

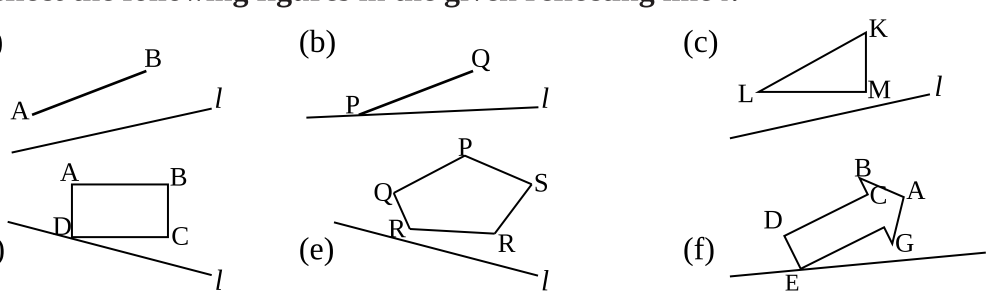
   (d) Define reflection.

   (e) Write any one property of reflection.

   (f) What will be the coordinates of the image of a point A(a, b) when it is reflected on the line x = 0?

3. **Reflect the following figures in the given reflecting line *l*:**

   (a) (b) (c) (d) (e) (f)



4. **Find the coordinates of the image of a point (4, –2) under the following reflecting axes:**

   (a) x-axis (b) line x = 0 (c) line y = x

   (d) line x + y = 0 (e) line x = 3 (f) line y – 2 = 0

5. **Find the coordinates of the image of the following points after reflection in y-axis:**

   (a) (2, 3) (b) (–1, 4) (c) (5, –6)

6. **Find the coordinates of the image of the following points after reflection in x = y:**

   (a) (–3, –5) (b) (0, –1) (c) (–6, 0)

7. **Find the coordinates of the image of the line segment joining the following points when reflect in the line x + 2 = 0:**

(a) (3, –5) and (1, 2) (b) (2, –1) and (0, 2) (c) (2, 0) and (–2, – 4)

8. (a) If A'(1, 3) is the image of the point A under reflection in x-axis, find the coordinates of the point A.

(b) Find the coordinates of the point P which reflects into P'(–5, 7) in the line x = 0.

9. (a) If P'(3a, – 3) is the image of the point P(7, b – 5) under the reflection in the line x = 0, find the values of a and b.

(b) If the image of P(2p – q, 5) is P'(2, p + q) under the reflection on the line y – x = 0, find the values of p and q.

10. (a) Find the reflecting axis when the point A(–2, –1) reflects into A'(–2, 1).

(b) Find the reflecting axis when the point A(3, –2) reflects into A'(–2, 3).

11. (a) Reflect the vertices of ΔABC having the vertices A(1, 2), B(0, –2) and C(2, 3) in the line y = 0 and then write the coordinates of the image ΔA'B'C'. Represent the above reflection on the same graph.

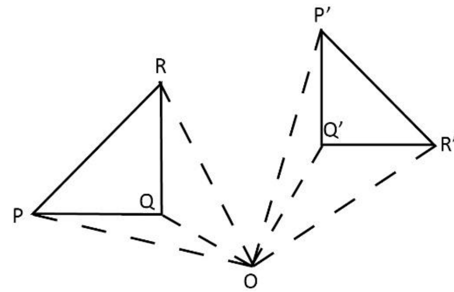
(b) Reflect the vertices of ΔKMN having the vertices K(1, 2), M(0, –2) and N(2, 3) in the line y = – x in the same graph and then write the coordinates of the image ΔK'M'N'.

12. (a) Find the coordinates of the vertices of the image of a quadrilateral PQRS with the vertices P(–1, 3), Q(–2, 5), R(–4, 1) and S(–5, 4) under the reflection in the line x = – 1. Draw this reflection on the graph.

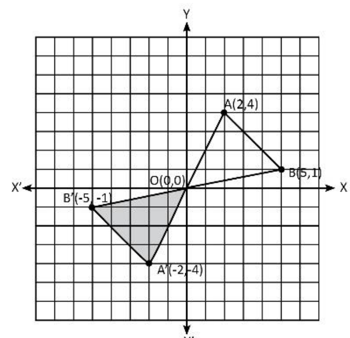
(b) The points P(–2, 3), A(0, 2), R(2, 3) and L(0, 4) are the vertices of the parallelogram PARL. Reflect the vertices of the parallelogram PARL in the line y + 3 = 0 by using graph and write down the coordinates of the vertices of the image parallelogram P'A'R'L'.

7.2 Rotation

A transformation in which each point of an object moves about a fixed point through an angle at constant distance and in the given direction is called a ***rotation***.



In the figure, ΔPQR moves itself through an angle θ about a fixed point O at the same distance OP = OP', OQ = OQ' and OR = OR' and its final image ΔP'Q'R' is also same as the initial object ΔPQR. The fixed point is called ***centre*** and the angle is called the ***angle of rotation***. The distance of the object from the centre is always equal. This constant distance is called the *radius of rotation*.



Properties of Rotation

The properties of rotation are:

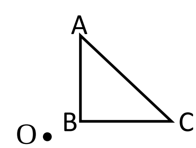
(i) Centre of rotation is an invariant point. In the adjoining rotation of ΔAOB, O is an invariant point.

(ii) Each point of object and image lie equidistant from the centre of rotation. In the adjoining rotation of ΔABC with centre O, OA = OA' and OB = OB'.

(iii) Object and image under rotation are congruent i.e. ΔAOB ≅ ΔA'OB'.

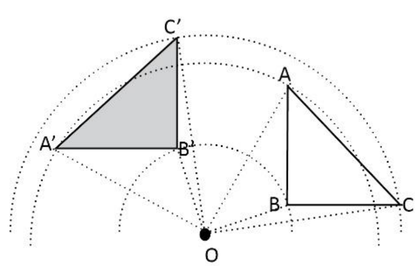
(iv) The perpendicular bisector of the segment joining any point and its image passes through the centre of rotation.

Example 1



Rotate the adjoining triangle ABC through 90° about the given point O in anticlockwise direction.

Solution: Here,



To rotate the given triangle ABC through 90° about the point O in anticlockwise direction:

(i) Join OA by dotted line.

(ii) Draw an arc or circle taking O as centre and OA as radius as shown in figure.

(iii) At O make an angle of 90° with compass or protractor in anticlockwise direction.

(iv) Join OA' by dotted line to meet the arc and circle at A'.

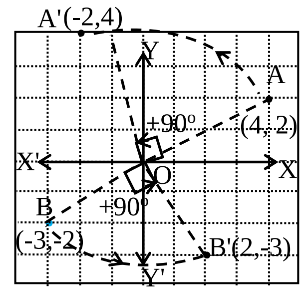
(v) Similarly, find the images B' and C' of points B and C.

(vi) Join A', B' and C'.

Hence, triangle A'B'C' be the image of triangle ABC under the rotation through an angle of 90° about centre O in anticlockwise direction.

Rotation in Cartesian Plane

(i) Rotation through 90° about origin

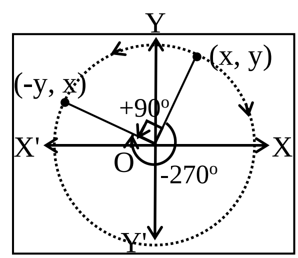


Discuss the rotation about origin in anticlockwise direction in the given adjoining graph.

In the graph, the points A(4, 2) and B(–3, –2) are rotated about the centre O(0, 0) through 90° in anticlockwise direction (i.e. O; +90°). We obtain the images A'(–2, 4) and B'(2, –3) respectively.

i.e. A(4, 2) $\xrightarrow{R[O;\,+90^\circ]}$ A'(–2, 4) and B(–3, –2) $\xrightarrow{R[O;\,+90^\circ]}$ B'(2, –3)

Hence, while rotating any point (x, y) through +90° about the origin, the position of x and y coordinates are interchanged and also sign changed in the x-coordinate of image only.



i.e., (x, y) $\xrightarrow{R[O;\,+90^\circ]}$ (–y, x)

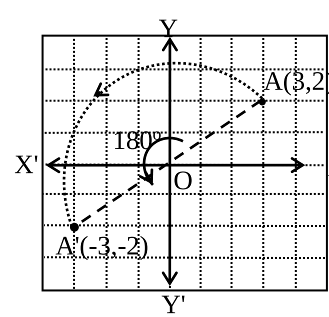
This type of rotation is called ***positive quarter turn***.

Again, rotation of the point (x, y) through 270° in clockwise direction about origin is same as the rotation through +90° about the origin.

i.e. their images are same.

∴ (x, y) $\xrightarrow{R[O;\,-270^\circ]}$ (–y, x).

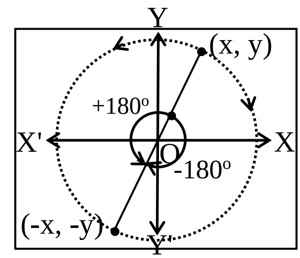
(ii) Rotation through 180° about origin



In the adjoining graph, the point A(3, 2) is rotated about the origin through 180° in anticlockwise direction (i.e., +180°). We obtain the image A'(–3, –2).

i.e., A(3, 2) $\xrightarrow{R[O;\,+180^\circ]}$ A'(–3, –2)

This shows that, while rotating any point (x, y) through an angle of +180° about the origin the coordinates remain the same with sign changed in the coordinates of image.



i.e., (x, y) $\xrightarrow{R[O;\,+180^\circ]}$ (–x, –y)

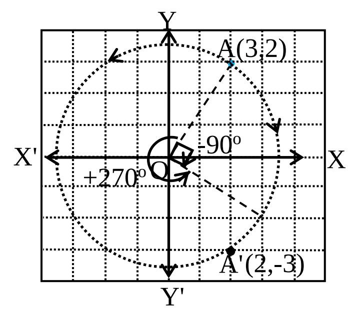
This type of rotation is called half turn about the origin.

Again, rotation of the point (x, y) through an angle of 180° about the origin in clockwise direction same as the rotation through 180° about the origin. That is their images are same.

So, (x, y) $\xrightarrow{R[O;\,-180^\circ]}$ (-x, -y).

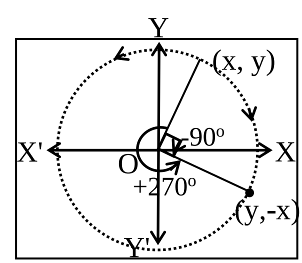
(iii) Rotation through 270° about origin

In the adjoining graph, the point A(3, 2) is rotated about the origin through 270° in anticlockwise direction (i.e. O; +270°). We obtain the image A'(2, –3).



i.e., A(3, 2) $\xrightarrow{\boldsymbol{R[O;\,+270^\circ]}}$ A'(2, –3)

This shows that, while rotating any point through an angle of +270° about the origin, the coordinates change the position with sign changed in y-coordinate of image.

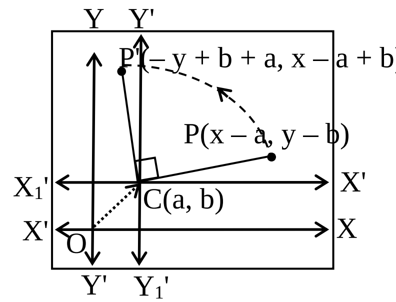


i.e., (x, y) $\xrightarrow{R[O;\,+270^\circ]}$ (y, -x)

Again, rotation of the point (x, y) through 90° in clockwise direction about origin (negative quarter turn) is same as the rotation through +270° about origin. i.e there images are same. So, (x, y) $\xrightarrow{R[O;\,-90^\circ]}$ (y, -x)

Note: When a point is rotated through 360° about the any point, it remains unchanged. This is called ***full turn***.

(iv) Rotation through + 90° about a point (a, b)

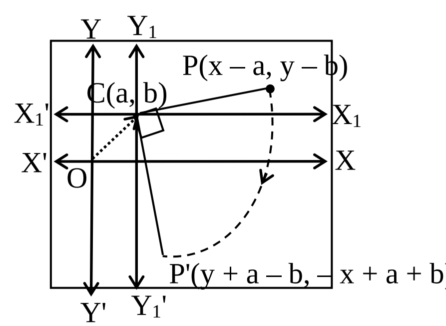


Let P(x, y) be a point and C(a, b) is the centre of rotation. Draw new axes as X_1CX_1' and Y_1CY_1' with the origin as C(a, b). Then the coordinates of P will be (x – a, y – b) with respect to the origin C(a, b). Now, rotate P(x – a, y – b) through 90° about the origin as C(a, b), we get

$$P(x-a, y-b) \xrightarrow{R[O;\, +90^\circ]} P'(-y+b, x-a)$$

Again, the coordinates of P' with respect to the actual origin O(0, 0) will be (– y + b + a, x – a + b). Hence, the coordinates of the image of the point P(x, y) under the rotation through +90° about the centre C(a, b) are P'(– y + a + b, x – a + b).

(v) Rotation through – 90° about a point (a, b)

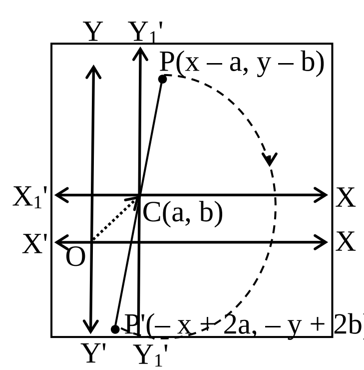


Let P(x, y) be a point and C(a, b) is the centre of rotation. Draw new axes as X_1CX_1' and Y_1CY_1' with the origin as C(a, b). Then the coordinates of P will be (x – a, y – b) with respect to the origin C(a, b). Now, rotate P(x – a, y – b) through – 90° about the origin as C(a, b), we get

$$P(x-a, y-b) \xrightarrow{R[O;\, -90^\circ]} P'(y-b, -x+a)$$

Again, the coordinates of P' with respect to the actual origin O(0, 0) will be (y + a – b, – x + a + b). Hence, the coordinates of the image of the point P(x, y) under the rotation through – 90° about the centre C(a, b) are P'(y + a – b, – x + a + b).

(vi) Rotation through 180° about a point (a, b)



Let P(x, y) be a point and C(a, b) is the centre of rotation. Draw new axes as X_1CX_1' and Y_1CY_1' with the origin as C(a, b). Then the coordinates of P will be (x – a, y – b) with respect to the origin C(a, b). Now, rotate P(x – a, y – b) through 180° about the origin as C(a, b), we get

$$P(x-a, y-b) \xrightarrow{R[O;\, 180^\circ]} P'(-x+a, -y+b)$$

Again, the coordinates of P' with respect to the actual origin O(0, 0) will be (– x + a + a, – y + b + b) = (– x + 2a, – y + 2b).

Hence, the coordinates of the image of the point P(x, y) under the rotation through 180° about the centre C(a, b) are P'(– x + 2a, – y + 2b).

Rules of Rotation in Cartesian Plane

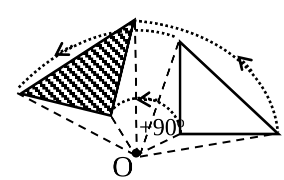
SN	Object	Centre of rotation	Angle of rotation	Image
1.	P(x, y)	(0, 0)	$\xrightarrow{+90° \text{ or } -270°}$	P'(– y, x)
2.	P(x, y)	(0, 0)	$\xrightarrow{-90° \text{ or } +270°}$	P'(y, – x)
3.	P(x, y)	(0, 0)	$\xrightarrow{\pm 180°}$	P'(– x, – y)
4.	P(x, y)	(a, b)	$\xrightarrow{+90° \text{ or } -270°}$	P'(– y + a + b, x – a + b)
5.	P(x, y)	(a, b)	$\xrightarrow{-90° \text{ or } +270°}$	P'(y + a – b, – x + a + b)
6.	P(x, y)	(a, b)	$\xrightarrow{\pm 180°}$	P'(2a – x, 2b – y)

Example 2

Find the image of the point (– 2, 4) under the rotation of negative quarter turn about the origin.

Solution: Here,

Rotating a point (– 2, 4) under negative quarter turn about the origin, we have



$$(x, y) \xrightarrow{R[O; -90^\circ]} (y, -x)$$

$$\therefore\ (-2, 4) \xrightarrow{R[O; -90^\circ]} (4, 2)$$

Hence, the image of a point (– 2, 4) under R[O; -90°] is (4, 2).

Example 3

If the image of a point (3a, b + 1) is (– 3, – 1) under the rotation through 180° about the origin, find the values of a and b.

Solution: Here,

We have, the image of a point (3a, b + 1) is (– 3a, – b – 1) under the rotation 180° about the origin.

But, by given, (3a, b + 1) $\xrightarrow{R[O;\, 180]}$ (– 3, – 1)

$\therefore$ (– 3a, – b – 1) = (– 3, – 1)

i.e., – 3a = – 3 and – b – 1 = – 1

or, a = 1 and b = 0.

Example 4

If the image of the point (1, –3) is (–3, –1) under the rotation about the origin, find the angle and direction of the rotation.

Solution: Here,

The image of a point (1, –3) is (–3, –1) under the rotation about the origin, in which the coordinates are interchanged and sign changed in y-coordinate only. This rotation is performed in the case of rotation through 270° in anti-clockwise direction (or 90° in clockwise direction) about the origin.

Example 5

A rectangle RECT has the vertices R(3, 0), E(6, 3), C(4, 5) and T(1, 2). Rotate the vertices of the rectangle RECT and write down the coordinates of the vertices of the image RECT when rotating through +90° about the origin. Represent the above transformation on the same graph.

Solution: Here,

The vertices of a rectangle RECT are R(3, 0), E(6, 3), C(4, 5) and T(1, 2).

Now, rotating the vertices of RECT about the origin through +90°, we have

(x, y) $\xrightarrow{R[O;\, +90^\circ]}$ (–y, x)

$\therefore$ R(3, 0) $\xrightarrow{R[O;\, +90^\circ]}$ R'(0, 3)

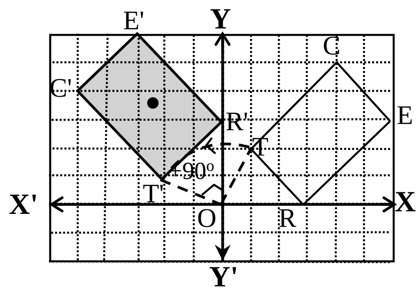
E(6, 3) $\xrightarrow{R[O;\, +90^\circ]}$ E'(–3, 6)

$$C(4, 5) \xrightarrow{R[O;\, +90^\circ]} C'(-5, 4)$$

$$T(1, 2) \xrightarrow{R[O;\, +90^\circ]} T'(-2, 1)$$

Hence, the coordinates of the vertices of the image R'E'C'T' are R'(0, 3), E'(–3, 6), C'(–5, 4) and T'(–2, 1).

Representing the above transformation on the same graph;



Example 6

Find the image of the point (– 2, 4) under the rotation of negative quarter turn about the center (1, 2).

Solution: Here,

Rotating a point (– 2, 4) under negative quarter turn about the center (1, 2), we have

$$(x, y) \xrightarrow{R[(a,b);\, -90^\circ]} (y + a - b, -x + a + b)$$

$$\therefore (-2, 4) \xrightarrow{R[O;\, -90^\circ]} (4 + 1 - 2, -(-2) + 1 + 2) = (3, 5)$$

Hence, the image of a point (– 2, 4) under R[(1, 2); – 90°] is (3, 5).

Exercise 7.2

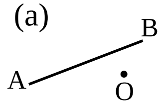
1. **Define the following terms:**
   (a) Rotation (b) Half-turn (c) Full Turn
   (d) Positive Quarter Turn (e) Negative Quarter Turn

2. (a) Write the co-ordinates of the image of a point K(a, b) when it is rotated around the origin through 90° in anti-clockwise direction.
   (b) Write down the images of the points P(a, b) and Q(c, d) when they are rotated through 270° in anti-clockwise direction about the centre (0, 0).

(c) What is the image of a point P(p, q) when it is rotated through 180° about the point (a, b) in clockwise direction?

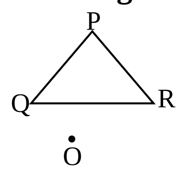
3. Rotate the following figures about the given point though the following instructions:

(a)



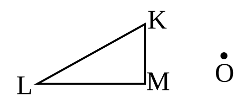
R[O; +90°]

(b)



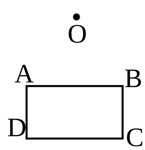
R[O; – 90°]

(c)



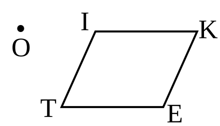
R[O; 180°]

(d)



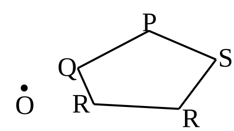
R[O; +270°]

(e)



R[O; – 270°]

(f)



R[O; – 180°]

4. Find the coordinates of the image of a point (3, –1) under the following rotations:

(a) R[O; + 90°] (b) R[O; 180°] (c) R[O; + 270°]

(d) R[O; – 270°] (e) R[O; – 180°] (f) R[O; – 90°]

5. Find the coordinates of the image of the following points under the positive quarter turn about the origin:

(a) (4, 6) (b) (2, 4) (c) (–5, –6)

6. Find the coordinates of the image of the following points under the negative quarter turn about the origin:

(a) (3, –5) (b) (– 5, 0) (c) (0, –4)

7. Find the coordinates of the image of the line segment joining the following points under the half turn about the origin:

(a) (3, –5) and (1, 2) (b) (2, –1) and (0, 2) (c) (2, 0) and (–2, – 4)

8. (a) If P'(–5, 4) is the image of the point P under the rotation about the origin through – 90°, find the coordinates of the point A.

(b) Find the coordinates of the point A which rotates into A'(–5, 7) under the rotation about the origin through + 90°.

9. (a) If P'(3a, b – 3) is the image of the point P(a + 2, 6) under the half turn about the origin, find the values of a and b.

(b) If the image of P(p – q, 5) is P'(2, p + q) under the negative quarter turn about the origin, find the values of p and q.

10. (a) Find the reflecting axis when the point A(2, –1) reflects into A'(–1, –2).

(b) Find the reflecting axis when the point P(3, –2) reflects into P'(–3, 2).

11. (a) Rotate ΔPQR having the vertices P(1, 3), Q(0, –4) and R(2, 2) under the rotation about the origin through 90° in clockwise direction and then write the coordinates of the vertices of the image ΔP'Q'R'. Represent the above reflection on the same graph.

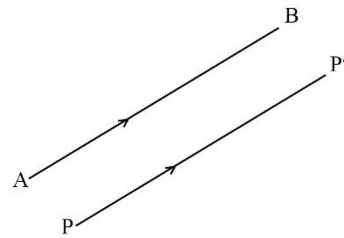
(b) Rotate ΔABC having the vertices A(3, 2), B(1, –2) and C(0, 3) under the rotation about the origin through 270° in clockwise direction on the same graph and then write the coordinates of the image ΔA'B'C'.

12. (a) Find the coordinates of the vertices of the image of a quadrilateral PQRS with the vertices P(–1, 3), Q(–2, 5), R(–4, 1) and S(–5, 4) under the negative half turn about the origin. Draw this rotation on the graph.

(b) The points C(–2, 0), D(–3, 2), E(1, 3) and F(3, 4) are the vertices of the parallelogram CDEF. Rotate the vertices of the parallelogram CDEF under the positive three-quarter turn about the origin by using graph and write down the coordinates of the vertices of the image parallelogram C'D'E'F'.

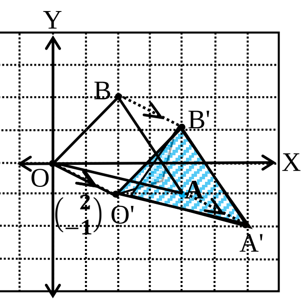
7.3 Translation

A transformation which maps an object to its images under the given vector is called a ***translation***.



In the figure, the point P is translated to its image P' along $\overrightarrow{AB}$. The vector $\overrightarrow{AB}$ on which the object moves into its image in the translation, is called the ***translation vector*** and $\overrightarrow{AB} = \overrightarrow{PP'}$, $\overrightarrow{AB} // \overrightarrow{PP'}$.

Properties of Translation



The properties of translation are:

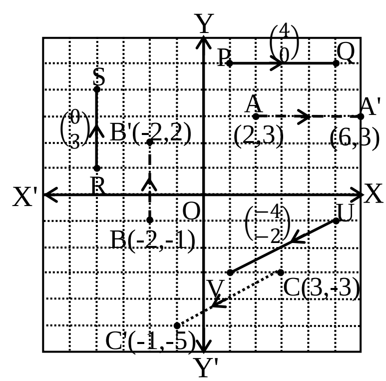
(i) The object and image are under translation are congruent. In the figure ΔAOB ≅ ΔA'B'C'T

(ii) Translation displaces each point in same distance and direction.

(iii) Translation is a direct isometric transformation

Translation in Cartesian Plane

Discuss the translation by given vector in the given adjoining graph.



In the graph, the point A(2, 3) moves to A'(6, 3) by the vector $\overrightarrow{PQ}$ with column vector $\begin{pmatrix}4\\0\end{pmatrix}$, which is moved 4 units right only. Similarly, the point B(–2, –1) moves to B'(–2, 2) by the vector $\overrightarrow{RS}$ with column vector $\begin{pmatrix}0\\3\end{pmatrix}$, which is moved 3 units upward only. Again, the point C(3, –3) moves to C'(–1, –5) by the vector $\overrightarrow{UV}$ with the column vector $\begin{pmatrix}-4\\-2\end{pmatrix}$ which is moved 4 units left and 2 units down.

Hence,

$$A(2, 3) \xrightarrow{\begin{pmatrix}4\\0\end{pmatrix}} A'(6, 3) = A'(2 + 4, 3 + 0).$$

$$B(-2, -1) \xrightarrow{\binom{0}{3}} B'(-2, 2) = B'(-2 + 0, -1 + 3).$$

$$C(3, -3) \xrightarrow{\binom{-4}{-2}} C'(-1, -5) = C'(3 - 4, -3 - 2).$$

Hence, the coordinates of the image of any point (x, y) under the translation by vector $\binom{a}{b}$ is (x + a, y + b).

i.e., $(x, y) \xrightarrow{\binom{a}{b}} (x + a, y + b)$.

Note: If (x', y') is the image of (x, y) by the translation vector T = $\binom{a}{b}$ then $T = \binom{a}{b} = \binom{x' - x}{y' - y}$.

Example 1

Translate the point P(3, – 4) by the translation vector $\binom{-4}{3}$. Write the coordinate of its image.

Solution:

Here, translating the point A(3, – 4) by $\binom{-4}{3}$, we get

$$A(3, -4) \xrightarrow{T\binom{-4}{3}} A'(3 - 4, -4 + 3) \quad [\because (x, y) \xrightarrow{T\binom{a}{b}} (x + a, y + b)]$$
$$= A'(-1, -1).$$

Hence, the coordinates of the image of the point A(3, – 4) is A'(– 1, – 1).

Example 2

If (4, 2b + 1) is the image of a point (a + 1, 2) by the translation vector $\binom{1}{3}$, find the values of a and b.

Solution: Here,

(4, 2b + 1) is the image of point (a + 1, 2).

Now, the image of point (a + 1, 2) by the translation vector $\binom{1}{3}$ is

$$(a + 1, 2) \xrightarrow{T\binom{1}{3}} (a + 1 + 1, 2 + 3) = (a + 2, 5)$$

By question, (4, 2b + 1) is the image of (a + 1, 2) by translation vector $\binom{1}{3}$.

So, (4, 2b + 1) = (a + 2, 5)

or, 4 = a + 2 and 2b + 1 = 5

∴ a = 2 and b = 2

Example 3

If the image of the point P(3, 2) is P'(–5, 1) under the translation, find the image of the point B(4, –2) under the same translation.

Solution: Here,

Let $\binom{a}{b}$ be a translation vector for the given translation then,

the image of the point P(3, 2) by $\binom{a}{b}$ is P' (3 + a, 2 + b).

But by question, the image of P(3, 2) is P'(– 5, 1).

∴ (3 + a, 2 + b) = (– 5, 1)

So, 3 + a = – 5 and 2 + b = 1

or, a = – 8 and b = – 1.

Hence, the translation vector is $\binom{-8}{-1}$.

Now, the image of the point B(4, –2) by the translation vector $\binom{-8}{-1}$ is

B(4, –2) $\xrightarrow{T\binom{-8}{-1}}$ B'(4 – 8, –2 – 1) = B'(–4, –3)

Example 4

Plot ΔKLM with the vertices K(1, 2), L(5, 1) and M(3, 4) on the graph paper and find the coordinates of the vertices of the image ΔK'L'M' of ΔKLM by the translation vector T = $\binom{2}{-1}$.

Solution

Plot the vertices K(1, 2), L(5, 1) and M(3, 4) of ΔABC on the graph paper.

Now, translating these vertices under the translation vector T = $\binom{2}{-1}$.

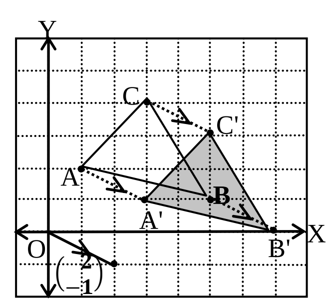
We obtain the following images:

(x, y) ⟶ (x + a, y + b)

∴ K(1, 2) ⟶ K'(1 + 2, 2 - 1) = K'(3, 1)

L(5, 1) ⟶ L'(5 + 2, 1 - 1) = L'(7, 0)

M(3, 4) ⟶M'(3 + 2, 4 - 1) = M'(5, 3)



Again, the image ΔA'B'C' is drawn on the same graph paper given alongside.

Exercise 7.3

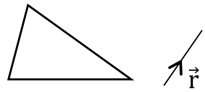
1. Answer the following questions in single sentence:

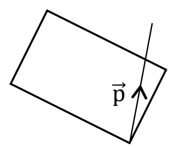
(a) What is translation?

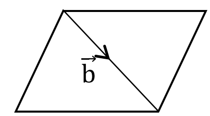
(b) Write any one property of translation.

(c) What is the image of a point T(p, q) under translation by the vector $\begin{pmatrix} c \\ d \end{pmatrix}$?

2. Draw the image of the following figures by the given translating vectors:

(a) 

(b) 

(c) 

3. Translate the following points by the given translating vector:

(a) A(4, –2) by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (b) B(0, 4) by $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ (c) C(–2, 4) by $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

4. (a) If (a, 2) is the image of the point (1, b + 3) by the translation vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, find the values of a and b.

(b) If (2p + 2, 4) is the image of the point (4, 3q + 1) by the translation vector $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$, find the values of p and q.

4. (a) If the image of the point (–2, 1) is (3, 3) by certain translation vector, find the translating vector.

(b) If the image of the point (4, –2) is (3, 5) by certain translation vector, find the coordinate of the image of the point (–5, 0) by the same translation vector.

11. (a) Plot the following vertices of the given geometric shape on the graph and find the coordinates of the vertices of its respective image under the translation vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Also, draw the given geometric shape and its image on the same graph.

(i) A(2, –1), B(–2, 0) and C(3, 3) of ΔABC.

(ii) P(1, –2), Q(4, –1), R(3, –4) and S(0, –4) of the quadrilateral PQRS.

(b) The vertices of the parallelogram KLMN are K(1, –2), L(5, –1), M(7, –3) and N(3, –4). Find the coordinates of the vertices of the image parallelogram K'L'M'N' under the translation by the vector $\begin{pmatrix}-1\\-3\end{pmatrix}$. Show the above transformation on the graph.

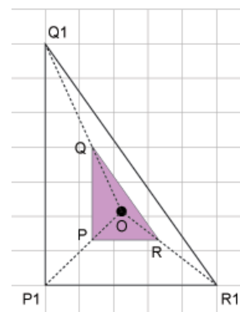
7.4 Enlargement

A transformation which transform a geometric figure about a fixed point O with certain value is called and enlargement. The fixed point is called centre of enlargement. The fixed value is called scale factor and is denoted by k.

In the figure, ΔABC is enlarged two times about the point O and its image becomes ΔA'B'C'. The point O is centre of enlargement and the number 2 is scale factor.

length of side in image = length of side in object × scale factor

Properties of Enlargement



The properties of enlargement are:

(i) Each side of the object is scaled by a scale factor.

(ii) The length of sides remain in the same proportion to each other.

(iii) Each line in the image is parallel to the corresponding line in the object.

(iv) All angles remain the same in the object and the image.

(v) The object and its image are similar.

(vi) If $k > 1$ then the image is larger than the object.

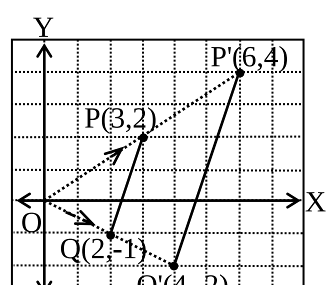
(vii) If $0 < k < 1$ then the image is smaller than the object.

(viii) If $k = 1$ then the image and the object are identical. This is an invariant.

(ix) If $k = -1$ then the image and the object are congruent but in opposite direction.

Enlargement in Cartesian Plane

Enlargement about origin by scale factor k



In the graph, the line segment PQ joining the points P(3, 2) and Q(2, -1) is enlarged at the centre O(0, 0) by scale factor 2. Then its image P'Q' has the point P'(6, 4) and Q'(4, -2).

i.e., P(3, 2) → P'(6, 4) = P'(2×3, 2×2) = P'(2(3, 2))

Q(2, –1)→ Q'(4, –2) = Q'(2×2, 2×(–1)) = Q'(2(2, –1))

Hence, the image of a point (x, y) under the enlargement

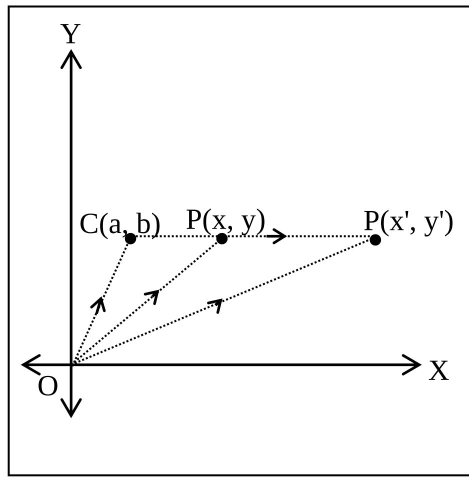
E[O; k] is k(x, y) = (kx, ky).

i.e., $(x, y) \xrightarrow{E[(0,\ 0);\ k]} (kx, ky)$

Enlargement about centre C(a, b) by scale factor k

In the adjoining figure, P'(x', y') is the image of P(x, y)

under E[(a, b); k], then $\overrightarrow{OP'} = \overrightarrow{OC} + \overrightarrow{CP'}$



or, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \overrightarrow{OC} + k\,\overrightarrow{CP} = \overrightarrow{OC} + k\,(\overrightarrow{OP} - \overrightarrow{OC})$

$= \begin{pmatrix} a \\ b \end{pmatrix} + k\left\{\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}\right\}$

$= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} kx \\ ky \end{pmatrix} - \begin{pmatrix} ka \\ kb \end{pmatrix}$

$= \begin{pmatrix} a + kx - ka \\ b + ky - kb \end{pmatrix}$

$= \begin{Bmatrix} k(x - a) + a \\ k(y - b) + b \end{Bmatrix}$

∴ x' = k(x – a) + a, y' = k(y – b) + b

Hence, the coordinates of P' are {k(x – a) + a, k(y – b) + b}.

Rules of Enlargement in Cartesian plane

SN	Object	Centre of enlargement	Scale factor	Image
1.	P(x, y)	(0, 0)	$\xrightarrow{k}$	P'(kx, ky)
2.	P(x, y)	(a, b)	$\xrightarrow{k}$	P'{k(x – a) + a, k(y – b) + b)}

Note:

(i) If k > 1, E[O; k] or E[(a, b); k] is enlargement.

(ii) If 0 < k < 1, E[O, k] or E[(a, b); k] is reduction.

(iii) If k = 1, E[O; k] or E[(a, b); k] is identity.

Example 1

Find the coordinates of the image of a point P(2, –4) under the enlargement with centre (0, 0) and scale factor 2.

Solution: Here,

Enlarging the point P(2, – 4) about the centre (0, 0) and scale factor 2, We have

$$(x, y) \xrightarrow{E[(0,0);\, k]} (kx, ky)$$

$$\therefore\ P(2, -4) \rightarrow P'(3 \times 2, 2 \times (-4)) = P'(6, -8)$$

Hence, the coordinates of the image point P' is (6, – 8).

Example 2

Enlarge the line segment AB joining the points A(4, – 6) and B(– 8, – 4) by E[(2, 1); 2].

Solution: Here,

Enlarging the line segment AB joining the points A(4, – 6) and B(– 8, – 4) by E[(2, 1); 2], we have

$$(x, y) \xrightarrow{E[(2,1);\, 2]} P'\{k(x - a) + a, k(y - b) + b)\}$$

$$\therefore\ A(4, -6) \xrightarrow{E[(2,1);\, 2]} A'\{2(4 - 2) + 2, 2(-6 - 1) + 1)\} = A'(6, -13)$$

$$B(-8, -4) \xrightarrow{E[(2,1);\, 2]} B'\{2(-8 - 2) + 2, 2(-4 - 1) + 1)\} = B'(-18, -9)$$

Example 3

If the point A(6, b) is enlarged about origin and scale factor 3 to its image A'(3a, 9), find the values of a and b.

Solution: Here,

$$A(6, b) \xrightarrow{E[O;\, 3]} A'(3a, 9)$$

But, we have $A(6, b) \xrightarrow{E[O;\, 3]} A'(6 \times 3, 3b) = (18, 3b)$

$\therefore$ (3a, 9) = (18, 3b)

i.e., 3a = 18 and 9 = 3b

or, a = 6 and b = 3.

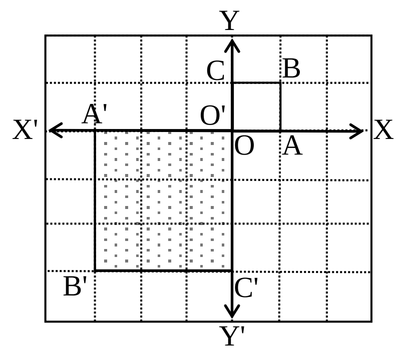
Example 4

Transform the square OABC having the vertices O(0, 0), A(1, 0), B(1, 1) and C(0, 1) by the enlargement E[(0, 0); –3]. Write down the coordinates of the vertices of the image square O'A'B'C' and represent the above transformation on the same graph.

Solution: Here,

The vertices of the square OABC are O(0, 0), A(1, 0), B(1, 1) and C(0, 1).

Now, enlarging the unit square OABC by E[(0, 0), –3], we get



O(0, 0) $\xrightarrow{E[(0,0); -3]}$ O'(–3×0, –3×0) = O'(0, 0)

A(1, 0) $\xrightarrow{E[(0,0); -3]}$ A'(–3×1, –3×0) = A'(–3, 0)

B(1, 1) $\xrightarrow{E[(0,0); -3]}$ B'(–3×1, –3×1) = B'(–3, –3)

C(0, 1) $\xrightarrow{E[(0,0); -3]}$ C'(–3×0, –3×1) = C'(0, –3)

Hence, the required coordinates of the image square O'A'B'C' are O'(0, 0), A'(–3, 0), B'(–3, –3) and C'(0, –3).

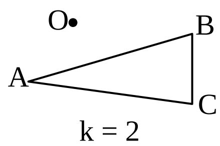
Square OABC and its image O'A'B'C' under enlargement E[(0, 0), –3] is given on the adjoining graph.

Exercise 7.4

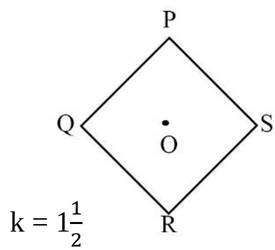
1. **Write the answer of the following questions in one sentence:**
   (a) What is enlargement?
   (b) If k = 1 in an enlargement, what type of image forms?
   (c) Define reduction.
   (d) When 0 < k < 1 in an enlargement, what type of image forms?
   (e) Write down the coordinates of the image of a point A(p, q) under E[(a, b); k].

2. Transform the following figures about given point and scale factor:

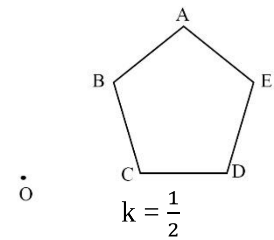
(a)



(b)



(c)



3. Enlarge the following points by the enlargement given below:

(a) P(–3, 1) by E[O; 1] (b) R(3, –3) by E[(0, 0); $\frac{1}{3}$]

(c) S(3, 0) by E(0, 0); -2]

4. Enlarge the following points by the given enlargement below:

(a) P(–2, 1) by E[(2, 1); 2] (b) R(2, –4) by E[(2, –1); $\frac{2}{3}$]

(c) S(2, 1) by E(–3, 2); –3]

5. (a) If the image of (3x, 2y) under the enlargement with centre as origin and scale factor 2 is (18, 8), find the values of x and y.

(b) If P'(3p – 2, q + 1) is the image of the point P(4, 1) under the enlargement E[(1, – 2); 2], find the values of p and q.

6. (a) If the image of P(6, 8) is P'(3, 4) under the enlargement with the centre (0, 0), find the coordinates of the image of the point (–2, –1) under the same enlargement.

(b) The point A(–2, 6) maps with A'(3, –1) under the enlargement with centre (2, 3). Find the scale factor of the enlargement.

7. (a) Enlarge a ΔABC having vertices A(2, 1), B(–2, 1) and C(–3, 1) from the origin with scale factor 2. Write down the coordinates of the vertices of the image of ΔLMN and represent the above transformation in the graph.

(b) The vertices of a unit square ASOK are A(0, 0), S(1, 0), O(1, 1) and K(0, 1). Write the coordinates of the image of the square ASOK under the enlargement with centre as origin and scale factor $1\frac{1}{2}$. Show the above enlargement on the same graph.

8. (a) P(2, 0), Q(2, 1), R(–1, 4) and S(–3, 2) are the vertices of a quadrilateral PQRS. Find the coordinates of the vertices of the image P'Q'R'S' of the quadrilateral PQRS under the enlargement E[(1, 2); 2]. Draw the square PQRS and its image on the same graph.

(b) A kite KITE having the vertices K(–3, 1), I(1, 3), T(–1, 2) and E(0, 1) is enlarged by E[(–2, 3); –3] on the graph. Write the coordinates of the vertices of the image of KITE.

Unit 8

Statistics

8.0 Review on Statistics

What is data? How many types of data are there?

What are frequency and cumulative frequency of the discrete data?

What is the arithmetic mean of the data?

What is the mean, median and mode of a data?

8.1 Partition Values of Ungrouped Data

What is the meaning of partition?

The values that divides the given data or series or observation into more than two parts is called partition values. There are generally three types of partition values. They are;

(i) Quartiles (ii) Deciles (iii) Percentiles

(i) Quartiles

Consider a data as

25, 28, 34, 56, 65, 78, 85.

By which terms do the data divide into four equal parts?

The terms 28, 56 and 78 divide the given data into four equal parts. These terms are called quartiles. The second term 28 is called first quartile (Q_1), the fourth term 56 is called second quartile (Q_2) and the sixth term 78 is called third quartile (Q_3). But, how can we find the items easily?

For this, we use the following formulae for individual and discrete data:

First Quartile (Q_1) = $\left(\frac{N+1}{4}\right)^{th}$ item

Second Quartile (Q_2) = $\left(\frac{2(N+1)}{4}\right)^{th}$ item = $\left(\frac{N+1}{2}\right)^{th}$ item = Median (Md)

Third Quartile (Q_3) = $\left(\frac{3(N+1)}{4}\right)^{th}$ item

(ii) Deciles

In a similar way as quartiles, deciles are those values that divide any set of a given observation into a total of ten equal parts. Therefore, there are a total of nine deciles. They are D_1, D_2, D_3, D_4, ……… D_9.

D_1 is the positional value for which one-tenth $(\frac{1}{10})$ of any given observation is either less or equal to D_1. However, the remaining nine-tenths $(\frac{9}{10})$ of the same observation is either greater than or equal to the value of D_1.

To calculate the deciles, we use the following formulae for individual and discrete data:

Value of Decile (D_n) = $\frac{n(N+1)^{th}}{10}$ item,

where n = 1, 2, 3, ..., 9 and N = Total no. of terms or sum of frequency.

(iii) Percentiles

In a similar way as quartiles and deciles, percentiles are those values that divide any set of a given observation into a total of hundred equal parts. Therefore, there are a total of ninety-nine percentiles. They are P_1, P_2, P_3, P_4, ……, P_{99}.

To calculate the deciles, we use the following formulae for **individual and discrete series**:

Value of Percentile (P_n) = $\frac{n(N+1)^{th}}{100}$ item,

where n = 1, 2, 3, ..., 99 and N = Total no. of terms.

Example 1

Find the first and third quartiles from the given data:

22, 26, 14, 30, 18, 17, 35, 41, 12, 32, 34

Solution: Here,

Arranging the given data in ascending order, we get

12, 14, 17, 18, 22, 26, 30, 32, 34, 35, 41

The number of items (N) = 11

Now, we have

First quartile (Q_1) = value of $\left(\frac{N+1}{4}\right)^{th}$ item

$$= \text{value of } \left(\frac{11+1}{4}\right)^{th} \text{ item}$$

$$= \text{value of } \left(\frac{12}{4}\right)^{th} \text{ item}$$

$$= \text{value of } 3^{rd} \text{ item}$$

$$= 17$$

Third quartile (Q_3) $= \text{value of } \left(\frac{3(N+1)}{4}\right)^{th} \text{ item}$

$$= \text{value of } \left(\frac{3(11+1)}{4}\right)^{th} \text{ item}$$

$$= \text{value of } \left(\frac{3\times 12}{4}\right)^{th} \text{ item}$$

$$= \text{value of } 9^{th} \text{ item}$$

$$= 34$$

Example 2

Find the first and third quartiles from the given data:

Marks (X)	50	60	75	82	91	90
No. of Students (f)	1	3	8	5	6	4

Solution:

Arranging the given data in ascending order,

Marks (X)	No. of Students (f)	Cumulative Frequency (c.f.)
50	1	1
60	3	4
75	8	12
82	5	17
90	6	23
91	4	27
	N = 27	

Now, we have

First quartile (Q_1) = $\left(\frac{N+1}{4}\right)^{th}$ item = $\left(\frac{27+1}{4}\right)^{th}$ item = $\left(\frac{28}{4}\right)^{th}$ item = 7^{th} item

In c.f. table the cumulative frequency just greater than 7 is 12. So, the corresponding value i.e. 75 is the first quartile.

Therefore, Q_1 = 75

Third quartile (Q_3) = $\frac{3(27+1)^{th}}{4}$ item = $\frac{3(27+1)^{th}}{4}$ item = $\left(\frac{84}{4}\right)^{th}$ item = 21^{th} item

In c.f. table the cumulative frequency just greater than 21 is 23. So, the corresponding value i.e. 90 is the third quartile.

Therefore, Q_3 = 90

Example 3

Calculate the 7^{th} deciles and 45^{th} percentile from the given data:

29, 31, 45, 27, 38, 59, 30, 28, 40, 25, 52

Solution: Here,

Arranging the given data in ascending order, we get

25, 27, 28, 29, 30 31, 38, 40, 45, 52, 59

Number of terms (N) = 11

Now, we have

7^{th} deciles (D_7) = $\frac{7(N+1)^{th}}{10}$ item

$= \frac{7(11+1)^{th}}{10}$ item

$= 8.4^{th}$ item

$= 8^{th}$ item + 0.4 (9^{th} item – 8^{th} item)

= 40 + 0.4 (45 – 40)

= 40 + 0.4 × 5 = 42

Again, we have

45^{th} percentile (P_{45}) $= \frac{45(N+1)^{th}}{100}$ item

$= \frac{45(11+1)^{th}}{100}$ item

$= 5.4^{th}$ item

= 5^{th} item + 0.4 (6^{th} item – 5^{th} item)

= 30 + 0.4 (31 – 30)

= 30 + 0.4 × 1 = 30.4

Example 4

Find the values of the fourth decile and sixty-sixth percentile from the given data:

Weight (in kg) (X)	10	15	20	25	30	35	40
No. of people (f)	7	5	2	4	3	3	6

Solution: Here,

Weight (in kg)	No. of people	Cumulative Freq. (cf)
10	7	7
15	5	12
20	2	14
25	4	18
30	3	21
35	3	24
40	6	30
	N = 30	

Here, $\Sigma f = N = 30$

Now

The value of fourth deciles (D_4) = Value of $\frac{4(N+1)^{th}}{10}$ item

= Value of $\frac{4(30+1)^{th}}{10}$ item

= Value of 12.4^{th} item

= 20 [∵ 12 < 12.4 < 14]

Again

The value of sixty-sixth percentile (P_{66}) = Value of $\frac{66(N+1)^{th}}{100}$ item

= Value of $\frac{66(30+1)^{th}}{100}$ item

= Value of 20.46th item

= 30 [∵ 18 < 20.46 < 34]

Exercise 8.1

1. Answer of the following questions in single sentence.

(a) What is individual data?
(b) What is discrete data?
(c) What do you mean by quartiles?
(d) How many deciles are there in a data set?
(e) Write the formula to calculate the 7th decile.
(f) What is the value of percentile?
(g) Write the formula to calculate the 51st percentile.

2. (a) Calculate the first quartile from the given data: 25, 26, 31, 42, 46, 52, 60

(b) Calculate the upper quartile from the given data:

21, 56, 48, 97, 53, 21, 54, 89, 75, 62, 14, 59, 80, 25, 41

3. (a) Find the lower quartile from the data given below:

X	10	20	30	40	50
f	4	6	8	7	10

(b) Find the third quartile from the given data:

Marks	5	10	15	20	25
No. of Students	8	12	9	5	6

4. (a) Calculate the fourth deciles from the data given below:

22, 25, 30, 42, 45, 50, 55, 60, 67

(b) Find the sixth deciles from the given data:

62, 14, 59, 56, 48, 97, 80, 75, 25, 21, 53, 21, 54, 89

5. (a) Find the third deciles from the following data:

X	3	6	9	12	15
f	2	4	1	3	2

(b) Calculate the eighth deciles from the following data:

Marks	12	14	16	18	20
No. of Students	12	25	24	42	30

6. (a) Calculate 40th percentile from the data given below:

23, 31, 35, 40, 51, 60, 68, 75, 80

(b) Find 60th percentile from the data given below:

20, 25, 36, 35, 48, 54, 52, 12, 13, 16, 41, 23, 41, 45, 60, 67, 46

7. (a) Find 15th percentile from the following data:

X	8	16	24	32	40
f	5	6	4	2	7

(b) Calculate 78th percentile from the given data:

Marks	10	20	30	40	50
No. of Students	5	6	7	2	4

8. Calculate the first, second and third quartiles from the data given below:

(a) 41, 23, 25, 28, 36, 56

(b) 12, 13, 16, 41, 23, 25, 28, 36, 39, 41, 45, 48, 54, 52, 50

9. (a) Calculate 3rd quartile, 6th deciles and 56 percentiles from the data given below:

23, 31, 35, 40, 51, 60, 68, 75, 80

(b) Find 1st quartile, 4th deciles and 77 percentiles from the given data:

20, 25, 36, 35, 48, 54, 52, 12, 13, 16, 41, 23, 41, 45, 60, 67, 46

10. Find the second and seventh deciles from the data given below:

(a)

X	5	15	25	35	45
f	5	6	7	4	8

(b)

Marks	12	16	18	25	32
No. of Students	6	4	7	5	3

11. Find 3rd quartile, 8th deciles and 95th percentile from the data given below:

(a)

X	5	15	25	35	45
f	5	6	7	4	8

(b)

Ages (in yrs.)	15	25	35	45	55
No. of people	12	25	55	43	57

(c) Collect the marks obtained by class 9 students in compulsory mathematics in unit test. Then arrange the data and discus with in classroom with following

i) the first quartile ii) the seventh deciles

iii) the fourth percentiles

8.2 Measures of Dispersion

Find the value of $\overline{X}$, M_d and mode of following data sets:

A: 20, 50, 80

B: 45, 50, 55

C: 5, 50, 95

Discuss in group that are all data sets same?

In above examples although the $\overline{X}$, M_d and mode are same, the nature of data sets are different. So measure of central tendency only given the average value. To find the scatterness of data we need further calculation. it is dispersion.

The measure of dispersion of statistical data measures the variability or spread of the data from the central value. It is the extent to which a distribution is stretched or squeezed. The measure of dispersion helps us to study the variability of the items. It is calculated by the difference of given observation and its central values either mean or median. Some measures of dispersions are the range, inter-quartile range, semi-inter-quartile range, mean deviation, variance and standard deviation, and their coefficients.

Dispersion is contrasted with location or central tendency, and together they are the most used properties of distributions.

Quartile Deviation

Quartile deviation is based on the lower quartile Q_1 and the upper quartile Q_3. The difference $Q_3 - Q_1$ is called the inter quartile range. The half of the difference $Q_3 - Q_1$ is called semi-inter-quartile range or the quartile deviation. Thus,

Quartile Deviation (QD) = $\frac{Q_3 - Q_1}{2}$

The quartile deviation is a slightly better measure of absolute dispersion than the range.

Coefficient of Quartile Deviation

A relative measure of dispersion based on the quartile deviation is called the coefficient of quartile deviation. It is a pure number free of any units of measurement. It can be used for comparing the dispersion of two or more data sets. It is defined as;

Coefficient of Quartile Deviation (CQD) = $\frac{Q_3 - Q_1}{Q_3 + Q_1}$

Exercise 8.2 (A)

1. Answer the following in single sentence.

(a) What do you mean by dispersion of a data set?

(b) Write different types of dispersions.

(c) Define quartile deviation and its coefficient.

(d) What is mean deviation of data set?

(e) What is the coefficient of mean deviation?

2. (a) If the lower and upper quartiles of a individual data are 23 and 46 respectively, find the quartile deviation and its coefficient.

(b) If the first and third quartiles of a discrete data are 12.5 and 51.2 respectively, calculate the quartile deviation and its coefficient.

(c) The quartile deviation of a data is 23 and its first quartile is 15. Compute its third quartile.

(d) The quartile deviation of a data is 45 and its third quartile is 50. Find its first quartile.

(e) If the coefficient of quartile deviation of a data is 0.45 and its first quartile is 25, find its third quartile.

(f) The coefficient of quartile deviation of a data is 0.62 and its third quartile is 26. calculate its first quartile.

3. (a) The quartile deviation and its coefficient of a data are 15.5 and 0.25 respectively, find its first and third quartiles.

(b) The quartile deviation and its coefficient of a data are 25.6 and 0.32 respectively, calculate its first and third quartiles.

4. Find the quartile deviation and its coefficient of the following data:

(a) Price of books (in Rs.): 45, 56, 67, 78, 70, 87, 90

(b) Ages of students (in yrs.): 12, 14, 13, 15, 16, 17

(c) Length of pencils (in cm): 15, 16, 19, 18, 20, 12, 15, 16, 21, 25, 24, 23

5. Calculate the quartile deviation and its coefficient of the data given below:

(a)

X	12	16	20	24	28
f	3	8	9	4	3

(b)

Marks	10	20	30	40	50
No. of students	9	5	7	16	21

(c)

Marks	25	35	45	55	65
No. of students	12	25	14	26	20

Mean Deviation

The mean deviation is the mean of the absolute deviations taken from the central tendency (mean, median, mode) of data. It is also called the mean absolute deviation. For a sample size N, the mean deviation is defined by;

Mean Deviation (MD) = $\frac{\sum|X - \bar{X}|}{N}$,

Where $\bar{X}$ is the central value of a data. The mean deviation is calculated either from mean or median, but only median is preferred because when the signs are ignored, the sum of deviation of the data taken from median is minimum.

To find he mean deviation, we use the following formulae.

For **individual series**:

Mean Deviation from Mean (MD) = $\frac{\sum|X-\bar{X}|}{N}$, where $\bar{X}$ is the mean of the data, and

Mean Deviation from Median (MD) = $\frac{\sum|X-M_d|}{N}$, where M_d is the median of the data.

And, for **discrete series**:

Mean Deviation from Mean (MD) = $\frac{\sum f|X-\bar{X}|}{N}$, where $\bar{X}$ is the mean of the data, and

Mean Deviation from Median (MD) = $\frac{\sum f|X-Md|}{N}$, where M_d is the median of the data.

Coefficient of Mean Deviation

The coefficient of mean deviation is calculated to compare the data of two series. The coefficient of mean deviation is calculated by dividing mean deviation by the average. If deviations are taken from mean, we divide it by mean, if the deviations are taken from median, then it is divided by median.

Coefficient of Mean Deviation from Mean (CMD) = $\frac{MD}{\bar{X}}$

Coefficient of Mean Deviation from Median (CMD) = $\frac{MD}{M_d}$

Example 1

Find the quartile deviation and its coefficient from the data given below:

22, 26, 14, 30, 18, 17, 35, 41, 12, 32, 34

Solution, Here,

Arranging the given data in ascending order, we get

12, 14, 17, 18, 22, 26, 30, 32, 34, 35, 41

The number of items (N) = 11

Now, we have

First quartile (Q_1) = value of $\left(\frac{N+1}{4}\right)^{th}$ item = value of $\left(\frac{11+1}{4}\right)^{th}$ item

= value of $\left(\frac{12}{4}\right)^{th}$ item = value of 3^{rd} item = 17.

Third quartile (Q_3) = value of $\left(\frac{3(N+1)}{4}\right)^{th}$ item

= value of $\left(\frac{3(11+1)}{4}\right)^{th}$ item

= value of $\left(\frac{3\times12}{4}\right)^{th}$ item

= value of 9^{rd} item

= 34.

$\therefore$ Quartile Deviation (QD) = $\frac{Q_3 - Q_1}{2} = \frac{34-17}{2} = \frac{17}{2} = 8.5$

$\therefore$ Coefficient of Quartile Deviation (CQD) = $\frac{Q3 - Q1}{Q3 + Q1} = \frac{34-17}{34+17} = \frac{17}{51} = \frac{1}{3}$

Example 2

Find the quartile deviation and its coefficient from the data given below:

Marks (X)	40	65	75	82	90	91
No. of Students (f)	16	12	8	5	3	1

Solution: Here,

Arranging the given data in ascending order,

Marks (X)	**No. of Students (f)**	**Cumulative Frequency (c.f.)**
40	6	6
65	12	18
75	18	36

82	5	41
90	3	44
91	1	45
	N = 45	

Now

First quartile (Q_1) $= \left(\frac{N+1}{4}\right)^{th}$ item $= \left(\frac{45+1}{4}\right)^{th}$ item

$= \left(\frac{46}{4}\right)^{th}$ item $= 11.5^{th}$ item

In c.f. table the cumulative frequency just greater than 11.5 is 18. so, the corresponding value i.e. 65 is the first quartile.

$\therefore Q_1 = 65$

Third quartile (Q_3) $= \left(\frac{3(N+1)}{4}\right)^{th}$ item $= \left(\frac{3(45+1)}{4}\right)^{th}$ item $= \left(\frac{138}{4}\right)^{th}$ item $= 34.5^{th}$ item

In c.f. table the cumulative frequency just greater than 34.5 is 36. So, the corresponding value i.e. 75 is the third quartile.

$\therefore Q_3 = 75$

$\therefore$ Quartile Deviation (QD) $= \frac{Q_3 - Q_1}{2} = \frac{75 - 65}{2} = \frac{10}{2} = 5.$

Coeff. Of QD $= \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{75 - 65}{75 + 65} = \frac{10}{140} = 0.07$

Example 3

Find the mean deviation from the median and its coefficient of the data given below: 40, 44, 54, 60, 62

Solution: Here,

Arranging the given data in ascending order, we get

40, 44, 54, 60, 62

The number of items (N) = 5

Median (M_d) $= \left(\frac{N + 1}{2}\right)$ th item $= \left(\frac{6 + 1}{2}\right)$ th item $= 3^{rd}$ term $= 54$

Now, making table to calculate MD from median,

X	$X - M_d$	$\|X - M_d\|$
40	– 14	14
44	– 10	10
54	0	0
60	6	6
62	8	8
N = 5		$\sum\|X - M_d\| = 38$

Now, we have

Mean Deviation from Median (MD) = $\frac{\sum|X - M_d|}{N} = \frac{38}{5} = 7.6$

Coefficient of Mean Deviation from Median (CMD) = $\frac{MD}{Md} = \frac{7.6}{54} = 0.14$

Example 4

Find the mean deviation from the mean and its coefficient of the data given below:

X	10	15	20	25	30
f	2	4	6	8	5

Solution:

Making table to calculate mean and MD from mean,

X	f	fX	$X - \bar{X}$	$\|X - \bar{X}\|$	$f\|X - \bar{X}\|$
10	2	20	-12	12	24
15	4	60	-7	7	28
20	6	120	-2	2	12

25	8	200	3	3	24
30	5	150	8	8	40
	N = 25	**$\sum f X = 550$**			**$\sum f \|x - \bar{X}\| = 128$**

$$\text{Mean } (\bar{X}) = \frac{\sum fx}{N} = \frac{550}{25} = 22$$

Now

$$\text{Mean deviation from Mean (MD)} = \frac{\sum f|X - \bar{X}|}{N} = \frac{128}{25} = 5.12$$

$$\text{Coefficient of Mean Deviation from Median (CMD)} = \frac{MD}{Mean} = \frac{5.12}{22} = 0.233$$

Exercise 8.2 (B)

1. **Find the mean deviation from mean and its coefficient of the data given below:**
   (a) Price of books (in Rs.): 43, 57, 65, 78, 73, 87, 90
   (b) Ages of students (in yrs.): 11, 15, 13, 18, 16, 17
   (c) Length of pencils (in cm): 13, 14, 19, 18, 23, 12, 15, 16, 21, 26, 24, 23
2. **Calculate the mean deviation from median and its coefficient from the following data:**
   (a) Price of bags (in Rs.): 45, 56, 67, 78, 70, 87, 90
   (b) Ages of students (in yrs.): 12, 14, 13, 15, 16, 17
   (c) Length of pencils (in cm): 15, 16, 19, 18, 20, 12, 15, 16, 21, 25, 24, 23
3. **Find the mean deviation from mean and its coefficient from the data given below:**

(a)

X	12	16	20	24	28
f	3	8	9	4	3

(b)

Marks	10	20	30	40	50
No. of students	9	5	7	16	21

(c)

Ages (in yrs)	25	35	45	55	65
No. of people	12	25	14	26	20

4. Calculate the mean deviation from median and its coefficient from the data given below:

(a)

X	5	15	25	35	45
f	5	9	12	7	8

(b)

Obtained Marks	10	20	30	40	50
No. of students	8	4	9	12	10

(c)

Marks	20	30	40	50	60
No. of students	10	2	8	9	4

5. Ask the age of 50 students of your school form grade 1 to 10 randomly. Construct discrete prequel distribution table and alunite mean deviation from mean and median of that data.

Standard Deviation

The standard deviation of a set of data is the square root of the average of the squared differences of the given observation from the Mean. The standard deviation (SD), also devoted by the lower case Greek letter sigma 'σ', is a measure that is used to quantify the amount of variation or dispersion of a data. A low standard deviation indicates that the data points tend to be close to the mean (also called the expected value) of the data, while a high standard deviation indicates that the data points are spread out over a wider range of values.

By definition of standard deviation, SD or σ = $\sqrt{\frac{\sum(X-\bar{X})^2}{N}}$

SD is also called the root-mean squared deviation.

Calculation of Standard Deviation

Standard deviation is calculated by using the following formulae:

(a) Standard Deviation of Individual series:

Let $X_1, X_2, X_3, \ldots\ldots, X_n$ are the variants then the SD is calculated by using any one of the following method:

(i) Actual Mean Method: SD (σ) = $\sqrt{\frac{\sum(X-\bar{X})^2}{N}} = \sqrt{\frac{\sum d^2}{N}}$, where $\boldsymbol{d} = \boldsymbol{X} - \bar{\boldsymbol{X}}$

(ii) Direct Method: SD (σ) = $\sqrt{\frac{\sum X^2}{N} - \left(\frac{\sum X}{N}\right)^2}$

(iii) Assumed Mean Method: SD (σ) = $\sqrt{\frac{\sum d^2}{N} - \left(\frac{\sum d}{N}\right)^2}$, where $\boldsymbol{d = X - A}$, $\boldsymbol{A}$ **=** Assumed Mean.

(b) Standard Deviation of discrete series;

Consider the discrete data series as;

X_1	X_2	X_3	…………	X_n
f_1	f_2	f_3	…………	f_n

Then the SD is calculated by using any one of the following metho;

(i) Actual Mean Method: SD (σ) = $\sqrt{\frac{\sum f(X-\bar{X})^2}{N}} = \sqrt{\frac{\sum fd^2}{N}}$, where $\boldsymbol{d} = \boldsymbol{X} - \bar{\boldsymbol{X}}$

(ii) Direct Method: SD (σ) = $\sqrt{\frac{\sum fX^2}{N} - \left(\frac{\sum fX}{N}\right)^2}$

(iii) Assumed Mean Method: SD (σ) = $\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$,

where $d = X - A$, A = Assumed Mean.

Coefficient of Standard Deviation

The standard deviation is the absolute measure of dispersion. Its relative measure is called the standard coefficient of dispersion or coefficient of standard deviation. It is defined as:

Coefficient of Standard Deviation (CSD) = $\frac{\text{SD}}{\bar{X}}$ or $\frac{\sigma}{\bar{X}}$

Variance

Variance is the square of the standard deviation of the data. That is the standard deviation is the square root of the variance of the data. Variance is the average of the squared deviations of given observations from the mean of a data.

i.e., Var = $\sigma^2 = \frac{\sum(X-\bar{X})^2}{N}$

Coefficient of Variation

The coefficient of variation (CV) is a measure of relative variability. It is the ratio of the standard deviation to the mean (average) expressed in percentage.

Coefficient of Variation (CV) = = $\frac{SD}{\bar{X}} \times 100\%$ or $\frac{\sigma}{\bar{X}} \times 100\%$

Example 1

Find the standard deviation and coefficient of variation from the data given below: 22, 25, 30, 35, 40, 45, 48

Solution: Here,

22, 25, 30, 35, 40, 45, 48

$$\text{Mean } (\bar{X}) = \frac{22 + 25 + 30 + 35 + 40 + 45 + 48}{7} = \frac{245}{7} = 35$$

Now, making table to calculate standard diviation:

X	**$X - \bar{X}$**	**$(X - \bar{X})^2$**
22	– 13	169
25	– 10	100
30	– 5	25
35	0	0
40	5	25
45	10	100
48	13	169
Total		$\sum(X - \bar{X})^2 = 588$

Now, we know that

$$\text{Standard Deviation (SD)} = \sqrt{\frac{\sum(X - \bar{X})^2}{N}} = \sqrt{\frac{588}{7}} = \sqrt{84} = 9.17$$

$$\text{Coefficient of Variation (CV)} = \frac{SD}{\bar{X}} \times 100\% = \frac{9.17}{35} \times 100\% = 26.2\%$$

Example 2

Find the standard deviation and its coefficient from the given data:

Marks	10	20	30	40	50
No. of Students	8	12	15	9	6

Solution: Here,

Making table to calculate the standard deviation.

Marks (X)	**No. of Stu. (*f*)**	***f* X**	**d = X – $\bar{X}$**	***f*d**	***f*d²**
10	8	80	– 18.6	– 148.8	2767.68
20	12	240	– 8.6	– 103.2	887.52
30	15	450	1.4	21.0	29.40
40	9	360	11.4	102.6	1169.64
50	6	300	21.4	128.4	2747.76
Total	N = 50	**$\sum f X = 1430$**		**$\sum fd = 0$**	**$\sum fd^2 = 7602$**

$$\text{Mean } (\bar{X}) = \frac{\sum f X}{\sum f} = \frac{1430}{50} = \frac{143}{5} = 28.6$$

Now, we know that

$$\text{SD or } \sigma = \sqrt{\frac{\sum f d^2}{N} - \left(\frac{\sum f d}{N}\right)^2} = \sqrt{\frac{7602}{50} - \left(\frac{0}{50}\right)^2}$$

$$= \sqrt{\frac{7602}{50}} = \sqrt{152.04}$$

$$= 12.33$$

Coefficient of SD (CSD) = $\frac{SD}{\bar{X}} = \frac{12.33}{28.6} = 0.43$

Exercise 8.2 (C)

1. **Write the answer of the following questions in one sentence.**

   (a) Define standard deviation. Write the formula of standard deviation for discrete data by assumed mean method.

   (b) What is the coefficient of standard deviation? Write its calculating formula.

   (c) What is variance? Write the formula to calculate the variance by direct method.

   (d) Define the coefficient of variation. Write the formula to calculate the coefficient of variation.

3. **Find the standard deviation and its coefficient of the given data:**

   (a) Obtained marks of students: 12, 19, 24, 28, 32, 36, 38, 45, 49

   (b) Length of foot of students (in cm): 15, 19, 21, 22, 25, 24, 26, 29, 25, 26, 24, 20, 18

   (c) Height of plants (in cm): 21, 22, 25, 26, 28, 31, 32, 36, 38, 42, 44, 45, 49

4. **Calculate the standard deviation and its coefficient of the data given below:**

(a)

X	15	24	30	56	60
f	5	6	8	9	7

(b)

Marks	50	60	70	80	90
No. of students	11	5	18	15	25

(c)

Marks	55	65	75	85	95	105
No. of students	9	15	8	12	24	10

5. Calculate the variance and the coefficient of variation of the data given below:

(a) Price of geometric boxes (in Rs.): 75, 85, 100, 125, 135, 160, 165, 175, 190

(b) Ages of children (in months.): 24, 36, 48, 49, 50, 55, 60, 68, 75

(c) Length of sticks (in cm): 15, 25, 28, 29, 31, 33, 35, 38, 42, 42, 44, 46, 52, 41, 45

6. Calculate the variance and the coefficient of variation of the data given below:

(a)

X	24	34	39	45	50
f	5	9	12	10	11

(b)

Marks	38	48	58	68	78
No. of students	12	9	10	13	15

7. Calculate the standard deviation and the coefficient of variation of the given data:

(a) 50, 60, 70, 80, 90, 100, 110, 120, 130, 10, 150

(b) 5, 15, 25, 35, 45, 55, 65, 75, 85, 95, 105, 115

8. Calculate the variance and the coefficient of variation of the following data:

(a)

X	12	16	20	24	28
f	3	8	9	4	3

(b)

Ages (in yrs)	25	35	45	55	65
No. of people	12	25	14	26	20

(c) Roll a dice for 100 times and record the occurrences of the numbers 1,2,3,4,5,6 in discrete data table. Find standard deviation and mean deviation and compare your findings with your classmates.

Answer

Exercise 1.1

1. Consult with your teacher.
2. (b) and (d)
3. (a) x = 5, y = 4 (b) x = 7 , y = 5

   (c) x = 5, y = - 2 (d) x = 7, y = 0 (e) x = 2, y = 1
4. (a) x = 2, y = 1 (b) x = 1, y = 1 (c) x = 5, y = 1 (d) x = 1, y = 1
5. Consult with your teacher.

Exercise 1.2

1. Consult with your teacher.
2. (a) {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

   (b) {(3, -2), (3, 3), (-2, 3), (-2, -2)}

   (c) {(1, 3), (1, -2), (2, 3), (2, -2), (3, 3), (3, -2)}

   (d) {(3, 1), (-2, 1), (3, 2), (-2, 2), (3, 3), (-2, 3)}
3. Consult with your teacher.
4. (a) {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}

   (b) {(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)}

   (c) No.
5. (a) n(A) = 2, n(B) = 3; 6, 6.

   (b) n(A) = 3, n(B) = 3, 9, 9
6. consult with teacher.
7. (a) A×B = {(1, -1), (1, 6), (2, -1), (2, 6), (3, -1), (3, 6), (4, -1), (4, 6)}

   B×A = {(-1.1), (6, 1), (-1, 2), (6, 2), (-1, 3), (6, 3), (-1, 4), (6, 4)}

   (b) P×Q = {(3, 0), (3, 3), (4, 0), (4, 3), (5, 0), (5, 3), (6, 0), (6, 3)}

   Q×P = {(0, 3), (3, 3), (0, 4), (3, 4), (0, 5), (3, 5), (0, 6), (3, 6)}

8. (a)

   (i) T×K = {(T, T), (T, O), (T, K), (T, Y), (O, T), (O, O), (O, K), (O, Y), (K, T), (K, O), (K, K), (K, Y), (Y, T), (Y, O), (Y, K), (Y, Y)}

K×T = {(T, T), (O, T), (K, T), (Y, T), (T, O), (O, O), (K, O), (Y, O), (T, K), (O, K), (K, K), (Y, K), (T, Y), (O, Y), (K, Y), (Y, Y)}.

(b) P = {r, a, m}, Q = {d, a, h, l}

(i) P×Q = {(r, d), (r, a), (r, h), (r, l), (a, d), (a, a), (a, h), (a, l), (m, d), (m, a), (m, h), (m, l)}

Q×P = {(d, r), (a, r), (h, r), (l, r), (d, a), (a, a), (h, a), (l, a), (d, m), (a, m), (h, m), (l, m)}

Exercise 1.3 (A)

1. Consult with your teacher.
2. a. R_1 = {(1, 5), (2, 4)}

   b. R_2 = {(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)}

   c. R_3 = {(2, 4)}

   d. verify with your teacher.
3. a. {(1, 3), (1, 6), (3, 6), (5, 6)} b. {(1, 1), (3, 3)} c. {(3, 6)}

   d. {(1, 1)} e. Consult with your teacher
4. a. R_1 = {(6, 2), (6, 4), (7, 2), (7, 4), (8, 2)}

   b. R_2 = {(6, 2), (6, 4), (6, 6), (7, 2), (7, 4), (7, 6), (8, 2), (8, 4), (8, 6), (10, 2), (10, 4), (10, 6)}

   c. consult with teacher

5 and 6. Consult with teacher

Exercise 1.3 (B)

1. Verify with your teacher.
2. a. D = {1, 2, 3}, R = {3, 5} b. D = {2, 3, 4}, R = {4, 6, 8, 12}

   c. D = {5, 6, 7, 8}, R = {8, 9, 10, 11}

   d. D = {8, 7, 6, 5, 4, 3}, R = {6, 5, 4, 3, 2, 1}
3. a. D = {3, 5}, R = {1, 2, 3} b. D = {4, 6, 8, 9, 12}, R = {2, 3, 4}

   c. D = {8, 9, 10, 11}, R = {5, 6, 7, 8} d. D = {6, 5, 4, 3, 2, 1}, R = {8, 7, 6, 5, 3, 4}
4. a. {(1, 5), (2, 6), (3, 6), (4, 7)} b. D = {1, 2, 3, 4}, R = {5, 6, 7}

   c. {(5, 1), (6, 2), (6, 3), (7, 4)}

5. a. R_1 = {2, 4, -6, -8} b. R = {3, 5, -5, -9}.
   c. R = {-3, -4, 1, 2} d. R = {2}
6. R = {(2, 4), (2, 6), (2, 8), (3, 6), (3, 9), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)}
   a. 2, 3, 4, 5, 6, 7, 8, 9} b. {4, 6, 8, 9, 2, 3, 5, 7}
   c. R^{-1} = {(4, 2), (6, 2), (8, 2), (6, 3), (9, 3), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)}
   d. D = {4, 6, 8, 9, 2, 3, 5, 7}, R = {2, 3, 4, 5, 6, 7, 8, 9}
7. a. {1, 3, 5, 7} b. {-1, 0, 3, 8} c. {0, 7, 20, 39}
   d. {25, 22, 19, 16, 13, 10}
8. a. {(0, 1), (2, 1), (2, 3), (3, 4)} b. {(-1, -1), (0, 0), (1, 1), (1, 2)}
   c. {(-1, 3), (-2, 4), (-3, 5), (-4, 6)} d. {(-2, 4), (2, 4), (-1, -1), (1, 1)}
9. a. {(1, 0), (3, 1), (5, 2), (7, 3)} b. {(-1, 0), (0, 1), (3, 2), (8, 3)}
   c. {(0, 1), (7, 2), (20, 3), (39, 4)}
   d. {(25, 0), (22,), (19, 2), (16, 3), (13, 4), (10, 5)}.

Exercise 1.4 (A)

1. Consult with teacher.
2. consult with teacher.
3. a) Function b) Function c) Function d) not a Function
   d) is not function other are.
4. Consult with your teacher
5. a) Function b) Not Function c) Not Function d) Function
6. Consult with your teacher.

Exercise 1.4 (B)

Consult with your teacher.

Exercise 1.4 (C)

1. a. one to one and onto b. one to one and onto
   c. Many to one and into d. many to one and onto
   e. Many to one and into f. many to one onto.
2. a. linear function b. Linear Function c. constant function

3. Consult with teacher

4. a. 6 b. 3 c. 0 d. 3

Exercise 1.4 (D)

1. (a) 13, 17, 25. (b) 1, -1, 7 (c) -1, 15, 55 (d) -1, -3, 6, -10.

4. (a) {-2, -4, 2} (b) {4, 10, 16}
   (c) {5, 2, -1, -4, -7} (d) {2, 3, 6}

5. (a) f(x) = 5x-18, f(5) = 7 (b) f(x) = 3x + 1, f(3) = 10.
   (c) f(x) = 2x + 9, f(-2) = 5 (d) f(x) = 4x-13, f(6) = 11

6. (a) h-5, x + h-5 (b) 0, h^2+ 2h, h+2
   (c) 0, $\frac{-2}{5}$, not defined (d) 25, 2, -9.

7. (a) D = {0, 1, 2, 3, 4, 5, 6, 7}, R = {0, 1, 2, 3, 4, 5, 6, 7} not a function.
   (b) {±5} (c) {0, 2} (d) Discuss and verify with teacher.

Exercise 1.5 (A)

1. Consult with your teacher
2. (a), (b), (d) are polynomials
3. (a) 1 (b) 1 (d) 3
4. (i) 2, y (ii) 2, *x* (iii) 3, y (iv) 3, x^2
   (v) 2/8, No literal coefficient
5. (a) 2 (b) 1 (c) 5 (d) 4 (e) 5 (f) 7
6. Consult with your teacher
7. Consult with your teacher
8. (a) a = 6, b = – 2, b) a= 0, b= 0 (c) a = 15, b = 16.
   (d) a = 9, $\sqrt[3]{8}$

Exercise 1.5 (B)

1. (a) $4x^3-2x^2 + 2x-7$ (b) $8x^3 + 3x^2-4$
   (c) $5x^4 + 4x^3 + 4x^2-6x + 15$ (d) $20x^4 + 8x^3-8x^2 + 7x-27$
2. (a) $2x^3-6x^2 + 8x-7$ (b) $6x^3 + 5x^2-6$
   (c) $5x^4-10x^3 + 6x-1$ (d) $-2x^4 + 8x^3 + 8x^2 + 7x-3$

4. (a) x^6-1 both (b) $x^3 + 1$ both

(c) $x^5 - 4x^4 + 9x^3 - 11x^2 + 6x - 4$ (d) $x^5 + 5x^4 - 13x^3 + 2x^2 + 10x - 5$

6. (a) $30x^5-48x^4-75x^3-6x^2-640x + 567$ (b) same as a

(c) $10x^4 + 20x^3-x^2 + 70x-126$ (d) $9x^3 + 6x^2 + 111x - 144$

7. Consult with your teacher.

8. a and b, $\frac{2}{3}x^3 + \frac{3}{2}x^2 + \frac{15}{9}x-1$ c and d, $\frac{2}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{9}x + 11$.

9. Consult with your teacher

10. (a) $(2x^2 + 6x-16)$ (b) $7x^2 + 10x-30$

(c) $-4x^3- 4x^2y -16xy^2$ (d) $-2x^3 + 3x^2y-5y^3$

Exercise 1.6 (A)

1) (a) 11, 13 (b) 4, 6 `(c) -2, -6 (d) 0, -5

(e) 80, 160 (f) 4, 2

2) and 3) consult with your teacher.

4) (a) 5, 8, 11, 14, 17 (b) 0, 3, 8, 15, 24

(c) 2, 4, 8, 16, 32 (d) -1, 4, -8, 16, -25.

5) (a) 1, 4, 9, 16, 25, 36 (b) 0, 3, 8, 15, 24, 35

(c) -1, 5, 15, 29, 47, 69

6) Consult with your teacher

Exercise 1.6 (B)

1. (a) 4, 7, 10, 13, 16 (b) 10, 17, 26, 37, 80.

(c) -2, 7, 22, 43, 70. (d) -2, 5, 24, 61, 122.

2. (a) tn = 2n + 3 (b) tn = 8-3n (c) tn = 4n + 3

(d) tn = n^2 + n (e) tn = $\left(\frac{3n-2}{3n+1}\right)$ (f) tn = $\left(\frac{3n-2}{n+6}\right)$

3. (a) tn = 5n-3 (b) tn = $4n^2$ (c) tn = 3n-1

(d) Pattern discuss with your teacher, tn = $\left(\frac{n(n+1)}{2}\right)^2$

(e) Pattern discuss with your teacher, tn = $\left(\frac{1}{2}\right)(n^2 + 3n + 6)$

4. Consult with your teacher.

Exercise 1.6 (C)

1. Consult with your teacher.
2. Sequences (a) and (b) series (c) and (d)
3. (a) $\sum_{n=1}^{7} 3n - 1$ (b) $(-1)^{n+1}\sum_{n=1}^{7} n$

   (c) $\sum_{n=1}^{14}(a-n)^n$ (d) $\sum_{n=1}^{10} \frac{n^2+5n+6}{4}$
4. (a) 18 (b) 26 (c) 90 (d) $\frac{2502}{945}$ (e) -6 (f) 36
5. (a) 45000 (b) 53973.125

Exercise 2.1

Show to your teacher.

Exercise 2.2

1. a) decreasing b) decreasing c) Zero
2. a) $\frac{10}{4} sqm.$ $\frac{10}{16}$ sq m b) $10, \frac{10}{4}, \frac{10}{16}, \frac{10}{64}, \frac{10}{256}, \frac{10}{1024},$ ………….. c) 0
3. a) Show to your teacher b) zero
4. Zero 5. $\frac{1}{2}$ 6. Show to your teacher.

Exercise 2.4

1. (a) 10 (b) -1 c) 31 d) 51
2. (a) $x \to 5$ (b) $x \to -4$ c) $a \to 10$ d) $a \to \infty$
3. Show to your teacher
4. (a) $\lim_{x\to 2} 2x = 4$ b) $\lim_{x\to 10} \frac{1}{x} = \frac{1}{10}$ c) $\lim_{x\to\infty} \frac{1}{x^2} = 0$ (d) $\lim_{x\to\infty} \frac{x+2}{2} = \infty$
5. *Present in the classroom.*

Exercise 3.1

1. Show it to your teacher.
2. a) 3 × 3 b) 3 × 2 c) 1 × 3 d) 3 × 1
3. a) 16 b) i = 2, j = 3, $a_{ij} = a_{23}$ c) u d) i = 2, j = 1
4. 3×2, $m_{3\times 2} = a_{ij}$, i = 3, j = 2
5. -2, 3, 7.
6. Consult to your teacher.

Exercise 3.2

1. Consult with your teacher.
2. (a) Square matrix (b) column matrix (c) rectangular matrix

   (d) row matrix (e) zero matrix (f) upper triangular matrix.
3. (a) $\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 4 & 9 \\ 4 & 16 & 36 \\ 9 & 36 & 81 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 & 4 \\ 1 & 0 & 1 \\ 4 & 1 & 0 \end{pmatrix}$
4. (a) a = 2, b = -2, c = 1, d = 3. (b), p = 4, q = 2

   (c) x = 2, y = 2, z = 1, w = 7. (d) x = 7, y = $\frac{3}{2}$, p = 2, q = -1

Exercise 3.3

1. Consult with your teacher.
2. a. M+N, N+M b. Q+R, R+Q c. Q+T, T+Q d. R+T, T+R

   e. P+U, U+P f. S+V, V+S
3. a) (i) $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 3 \end{pmatrix}$

   (iii) $\begin{pmatrix} a-3 & b-4 \\ c-7 & d-9 \end{pmatrix}$ (iv) $\begin{pmatrix} 3-a & 4-b \\ 7-c & g-d \end{pmatrix}$

   (v) $\begin{pmatrix} e+1 & 3-f & g-2 \\ 2-h & i+5 & j+3 \end{pmatrix}$ (vi) $\begin{pmatrix} e+1 & 3-f & g-2 \\ 2-h & i+5 & j+3 \end{pmatrix}$

   (vii) $\begin{pmatrix} e+1 & 3-f & g-2 \\ 2-h & i+5 & j+3 \end{pmatrix}$ (viii) $\begin{pmatrix} e+1 & 3-f & g-2 \\ 2-h & i+5 & j+3 \end{pmatrix}$

   b) –A = $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$
4. a. x = 2, y = 1 b. x = 5, y = 3 c. x = 2, y = 2, z = 10 d. x = 2, y = 1
5. a. (i) (9 21 4) (ii) (1 11 4) (iii) (5 16 -2)

   b. (i) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
6. a. (i) $\begin{pmatrix} 3 & 2 \\ 5 & -5 \end{pmatrix}$ (ii) $\begin{pmatrix} 12 & -1 \\ -6 & 9 \end{pmatrix}$ (iii) (4 4) (iv) (-12 -11)

   (e) $\begin{pmatrix} 4 \\ -5 \\ -3 \end{pmatrix}$ (f) $\begin{pmatrix} 5 \\ 0 \\ 10 \end{pmatrix}$
7. a. $\begin{pmatrix} -5 & -8 \\ 9 & -3 \end{pmatrix}$ b. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ c. $\begin{pmatrix} -10 & -16 \\ 18 & -6 \end{pmatrix}$

8. Consult with your teacher.

9. $\begin{pmatrix} 106 & 134 & 171 \\ 80 & 124 & 125 \\ 112 & 178 & 102 \\ 100 & 120 & 160 \end{pmatrix}$

Exercise 3.4

1. Consult with your teacher.

2. a. $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ b. (m n p) c. $\begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 5 \end{pmatrix}$

   d. $\begin{pmatrix} 3 & -2 \\ 5 & -7 \\ -9 & 4 \end{pmatrix}$ e. $\begin{pmatrix} a & d & g \\ b & c & h \\ c & f & i \end{pmatrix}$

3-7 consult with your teacher

Exercise 3.5

1. Consult with your teacher.

2. a) $\begin{pmatrix} 21 & -7 \\ 28 & 14 \end{pmatrix}$ b) $\begin{pmatrix} 9 & 7 \\ 14 & 16 \end{pmatrix}$ c) $\begin{pmatrix} 4 & 6 \\ 12 & 13 \end{pmatrix}$ d) $\begin{pmatrix} 28 & 20 \\ 60 & 48 \end{pmatrix}$

   e) $\begin{pmatrix} 28 & 20 \\ 60 & 48 \end{pmatrix}$ f) $\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$ g) $\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$ h) $\begin{pmatrix} 14 & 5 \\ 32 & 30 \end{pmatrix}$

   i) $\begin{pmatrix} 14 & 5 \\ 32 & 30 \end{pmatrix}$ j) $\begin{pmatrix} 15 & 16 \\ 80 & 68 \end{pmatrix}$ k) $\begin{pmatrix} 15 & 16 \\ 80 & 68 \end{pmatrix}$

3. a) i. (2 3) ii. $\begin{pmatrix} 1 & 5 \\ 3 & 15 \end{pmatrix}$ iii. (-12)

a) i. $\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$ ii. $\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$ iii. $\begin{pmatrix} 0 & 3 \\ 1 & -1 \end{pmatrix}$ iv. $\begin{pmatrix} -1 & 3 \\ 1 & 0 \end{pmatrix}$

   v. $\begin{pmatrix} 7 & 6 \\ 4 & 7 \end{pmatrix}$ vi. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ vii. $\begin{pmatrix} 5 & 10 \\ 6 & 3 \end{pmatrix}$ viii. $\begin{pmatrix} 7 & 6 \\ 4 & 7 \end{pmatrix}$

   c) MN = $\begin{pmatrix} -11 & -5 \\ -18 & -21 \end{pmatrix}$, NM = $\begin{pmatrix} -4 & 8 & 15 \\ 2 & 8 & 24 \\ -3 & -12 & -36 \end{pmatrix}$, MN≠M

   d) Consult with your teacher.

4. a. $\begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$ b. $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ c. x = 8, y = 126, z = -7/3 d. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   e. a = 2, b = 2 f. $x = \frac{3}{2}$, $y = 0$, $z = \frac{4}{3}$

5. Consult with your teacher.

6. a. $\frac{1}{1}$ b. (5 4) c. x = 2, y = 3 d. consult with your teacher

e. $x = \frac{3}{4}$, y = 0, z = $\frac{5}{4}$ f. and g. consult with your teacher

7. Consult with your teacher.

Exercise 4.1

1. (a) $x^2+y^2+8x-64$ (b) $y-5 = 0$ (c) $x+3 = 0$

(d) $x^2+y^2+10x+2y-7 = 0$ (e) $x^2+y^2-2x-12y-12 = 0$

2. (a) (1, 1) (b) (1, 3) (c) yes (i) yes (ii) No (iii) yes (iv) yes

(d) Show to your teacher (e) $\frac{15}{14}$ (f) (4, 4)

3. (a) $(x\text{-a})^{2}+(\text{y-b})^2 = \text{k}^2$

(b) (i) $x^2-4y+4 = 0$ (ii) $3x-5y = 1$ (iii) $3x-y = 5$

(iv) $7x+4y = 4$ (v) $x\text{-}y = 0$

4. Show to your teacher

Exercise 4.2

1. (a) 5 units (b) 64 units (c) $\sqrt{8}$ Units (d) $\sqrt{8}5$ units

2. (a) $\left(1,\frac{-4}{5}\right)$ (b) $\left(\frac{-7}{5},\frac{21}{5}\right)$ (c) $\left(\frac{11}{7},\frac{-1}{7}\right)$ (d) $\left(\frac{4}{5},\frac{4}{5}\right)$

3. (a) (–28, –23) (b) $\left(\frac{10}{3},5\right)$ (c) (–11, 33) (d) (-7)

4. (a) (6, 3) (b) (3, 5) (c) (–4, 1) (d) (1, 1) (e) (6, 1)

5. (a) 1: 6 (b) 2: 1(c) 4: 7(d) 1: $2\frac{7}{9}$ (e) 2: 5, 4

6. (a) (–5, –2), (–2, –4) and (1, –6) (b) (0, 7), (3, 5)

(c) Show to your teacher

7. (a) (–5, 0) and $\left(0,\frac{15}{2}\right)$(b) $\left(\frac{92}{3},0\right)$ and (0, 8) (c) Show to your teacher.

8. (a) Show to your teacher. (b) Parallelogram (c) (–8, 2)

(d) Show to your teacher (f) (8, 3) (g) Show to your teacher

9. (a) (i) (1, 4) (ii) (2, –3) (iii) $\left(\frac{10}{3},\frac{10}{3}\right)$

(b) (3, 1) (c) –5, –5 (d) $\sqrt{53}$ units (e) (4, 0), (0, 6)

Exercise 4.3

1. (a) $\frac{1}{\sqrt{3}}$ (b) 1 (c) 0 (d) $\sqrt{3}$
2. (a) 45^0 (b) 60^0 (c) 30^0 (d) 0^0
3. (a) 1 (b) $\frac{1}{2}$ (c) $\frac{5}{2}$ (d) $\frac{2}{3}$
4. (a) 3 (b) 7 (c) –2
5. (a) y = 0 (b) x = 0 (c) x = 3 (d) x = –2

(e) y = 5 (f) y = –4 (g) y = 2 (h) x = –3

6. Show to your teacher.
7. (a) a = 6, b = 6 (b) a = 10, b = 10

Exercise 4.4

1.(a) y = 5x + 3 (b) x - y = 4 (c) $2x$ – 3y + 12 = 0

(d) $x - \sqrt{3y} + 3 = 0$ (e) $\sqrt{3y} = y$

2. (a) $x - \sqrt{3y} + 4\sqrt{3} = 0$ (b) $\sqrt{3y} + y = 5$ (c) $x + y = 6$

(d) $3x + y = 4$, $x - y + 4 = 0$

3. (a) $3x - 4y = 12$ (b) $x - y + 3 = 0$

(c) $x + y = 5$ (d)$3x + 4y - 12 = 0$

4. (a) $x + y = 8$ (b) (i) $x + y = 10$ (ii) $x - y = 2$

(c) $x - 2y = 7$ (d) $4x + 3y = 12$, $16x + 3y = 24$

5. (a) $x + 2y = 6$ (b) $8x + 5y + 10 = 0$ (c) $9x - 20y + 96 = 0$
6. (a) $\sqrt{3x} + y = 4$ (b) $x + y = 6\sqrt{2}$ (c) $y = 5$

(d) $x - \sqrt{3y} + 6 = 0$ (e) $x + \sqrt{3y} = 14$ (f) $x + y = 4$

7. (a) $x + y = 6$ (b) $\sqrt{3y} + y = 6$ (c) $x + \sqrt{3y} = 5$

Exercise 4.5

1. (a) y = $-4x - 3$; -4, -3 (b) $y = \frac{8}{3}x - 2; \frac{8}{3}x - 2$

(c) $y = \frac{-5}{3}x + 3; \frac{-5}{3}, 3$ (d) $y = 4x + \frac{5}{3}$ (e) $y = 3x - \frac{3}{2}$

2. (a) $\frac{x}{4} - \frac{4}{3} = 1, 4 - 3$ (b) $\frac{x}{-3} + \frac{4}{3} = 1. -3, 3$

(c) $\frac{x}{5} + \frac{4}{5} = 1; 5, 5$ (d) $\frac{x}{-4} + \frac{y}{4} - 1, -4, 4$

(e) $\frac{x}{5} - \frac{y}{2} = 1, 5, -2$ (f) $\frac{x}{-9} + \frac{y}{24} = 1, -9, 27$

3. (a) $x\cos 60° + y \sin 60° = 2, 60°, 2$

(b) $x\cos 45° + y\sin 45° = \sqrt{2}\ 45°,$

(c) $x\cos 30° + y\sin 30° = 5, 300, 5$

(d) $x\text{cox}^{-1}\left(\frac{3}{4}\right) + y\sin^{-1}\left(\frac{3}{5}\right) = 5$

e) $x\cos 30° + y\sin 30° = 3, 30°, 3$

4. (a) $y = \sqrt{3}x - 6, \frac{x}{2/\sqrt{3}} + \frac{4}{6} = 1, x\cos 30° + y\sin 30° = 3$

(b) $y = \sqrt{3}x - \frac{11}{2\sqrt{3}}, \frac{x}{\frac{11}{2\sqrt{3}}} + \frac{y}{\frac{11}{2}} = 1; x\cos 30^0 + y\sin 30^0 = \frac{11}{4}$

(c) $y = \frac{-3}{2}x + \frac{1}{\sqrt{2}}; \frac{x}{3/\sqrt{2}} + \frac{y}{1/\sqrt{2}} = 1; x\cos^{-1}\left(\frac{4}{\sqrt{50}}\right) + y\sin^{-1}\left(\frac{6}{\sqrt{50}}\right) = \frac{3\sqrt{2}}{\sqrt{50}}$

(d)) $y = \frac{1}{\sqrt{3}}x - 2\sqrt{3}; \frac{x}{6} - \frac{y}{2\sqrt{3}} = 1;\ x\cos 30^0 - y\sin 30^0 = 3$

5. (a) 6sq. units (b) 14sq. unit (c) Show to your teacher

Exercise 4.6

1. (a) $\sqrt{3} - y + 5 - 3\sqrt{3} = 0$ (b) $x + \sqrt{3y} + 2 + 4\sqrt{3} = 0$

(c) $\sqrt{3}x + y - 2 + 5\sqrt{3} = 0$ (d) $x - y = 11$

(e) $x - \sqrt{3y} = 7 - 4\sqrt{3}$

2. (a) $2x - 7y + 29 = 0$ (b) $x + y = 1$ (c) $3x - 5y = 25$

(d) $2x - y = 0$ (e) $bx + ay = ab$

3. (a) Show to your teacher.

4. (a) $x + y = 9$ (b) Show to your teacher

(c) $x - 5y = 0$ (d) $3x - y = 0$

5. (a) $2x + y = 1,\ x - 2y = 3,\ x + 3y = 8$ (b) $2x - 3y = 13,\ \sqrt{13}$

(c) $x + y = 9, \sqrt{50}$ (d) Show to your teacher.

6. (a) a = 6 (b) b = 2 (c) k = 6 (d) p = 0

Exercise 4.7

1. (a) $\frac{3\sqrt{3}-8}{2}$ units (b) $\frac{7+5\sqrt{2}}{\sqrt{2}}$ units (c) 8 units (d) 1 units

   (e) 5 units (f) $\sqrt{p^2+q^2}$ (g) $\frac{1}{\sqrt{2}}$units (h) $\frac{17}{5}$ units

2. (a) 8 units (b) $\sqrt{34}$ units (c) $\frac{19}{2\sqrt{13}}$ units (d) 8 units

   (e) 4 units

3. (a) 20 or 6 (b) –6

Exercise 4.8

1. (a) 5sq. unit (b) 1 sq. unit (c) 8 sq. unit (d) 4.5 sq. units

   (e) 8 sq. unit (f) 10 sq. unit (g) 0 sq. unit (h) 7 sq. unit

   (i) 0 sq. unit (j) 10 sq. unit

2. Show to your teacher.

3. (a) 24 sq. unit (b) 2.5 sq. unit (c) 20.5 sq. unit

   (d) 41 sq. units (e) 44 sq. unit

4. Show to your teacher.

5. (a) $\frac{11}{8}$ (b) $\frac{8}{5}$ (c) –3 (d) Show to your teacher.

6. (a) (2, 4) 6 sq. units (b) (i) 18 sq unit and $\frac{9}{2}$ sq. unit.

Exercise 5.1

2. (a) $198030''$ b) $36925''$ c) $201388''$

3. (a)36.9 b) 25.29833 c) 48.84583

4. (a) $250029''$ b) $253429''$ c) $257499''$

5. (a).36.3^{g} b)27.283^{g} c)79.4723

6. (a) 38.29 b) 32.08905 c) 76.90090

7(a) 75.5 b) 46.917 c)85.4111

8. (a) 60.75π/180 b) 57.84084π/180 c) 67.572π/180

9. a) $22^{\cdot}30'$ b) $51^{\cdot}25'$ 42.8574 c) $16^{\cdot}52'30''$

10. a) $12^{9}50'$ b) $50'$ c) $47^{9}61'9''$

11. a) 80.1 b) 83.3 c) 104^g d) 79^g e) $\pi^c/4$

12. a) 107.1 b) 2π/9, 5π/18, π/2 c) 40^9, 80^g, 120^g, 160^g

d)10.5, 52.5 e) 25:27

13. a) 75^0 b) 97.5^0 c)19π/24

14. a) 254.78^g b) 0.037cm c) 49.49cm d) 33.33m

e) 4.365cm f)11.62cm

Exercise 5.2

2 a. cosθ = 4/5 tanθ = 3/4

b. sinA = 5/13, TanA = 5/12 CosecA = 13/5

c. sinθ = 15/17 cotθ = 8/15 cosecθ = 17/15

d. sinα = $\frac{2\sqrt{a}}{a+1}$ cosα = $\frac{a-1}{a+1}$ e. cos$x$ = $\frac{1}{\sqrt{2}}$ Tan$x$ = 1

3. a) $-\frac{213}{60}$ b) $-\frac{1}{5}$ c) $\frac{1}{18}$

4. a) $\frac{56}{65}$ b) $\frac{33}{65}$ c) $\frac{16}{65}$ d) $\frac{63}{65}$ e) $\frac{56}{33}$ f) $-\frac{16}{63}$

5. a. 2mn

Exercise 5.3

2. a) $\frac{1}{4}$ b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ c) $\frac{2\sqrt{3}+1}{2}$ d) $\frac{\sqrt{3}}{2}$ e) 2 f) $\frac{5}{4}$

g) 0 h) 3 i) $-\frac{\sqrt{2}}{\sqrt{6}+2}$ j) $\frac{13}{6}$

6. a) x = -4 b) 3 c) 3 d) 4 e) 3

Exercise 5.4

2. a. $\sin^2A - \sin^2B$ b. -2cosα c. $1- \cos^2\alpha$ d. $1 - \tan^4A$

e. $1 - \sin^4A$ f. $1 - \tan^4\alpha$

3. a. (tanA + sinA) (tanA – sinA) b. (cosA –SecA) (cos A + secA)

c. $\sin^2x(1 + \cos^2x)$ d. $(\sec^2\theta + \cos^2\theta)(\sec\theta + \cos\theta)(\sec\theta - \cos\theta)$

e. (sin$x$ + 2) (sin$x$ +3)

Exercise 5.5

2. a) $\frac{-\sqrt{3}}{2}$ b) $\frac{-1}{\sqrt{2}}$ c) –1 d) $\frac{-1}{\sqrt{2}}$ e) –1 f) –1 g) $\frac{2}{\sqrt{3}}$

3. a. $\frac{\text{sinA.cosA}-1}{\text{CosA}}$ b. $-\tan^2 A$

7. a. $-\frac{1}{2}$ b. $-\frac{7}{6}$ c. $-\frac{3}{2}$ d. $\frac{1}{2}$ e. $\frac{1}{2}$ f. $1+\sqrt{2}$

g. $\frac{7}{2}$ h. $-\frac{3}{2}$ i. 0 j. 2

8. a. – cosecA b. 1 c. tanA d) $-\sin\alpha$ e. $\sec^2\theta$ f. $\frac{1}{sin\alpha\times\cos\alpha}$

Exercise 5.6

2. (a) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (b) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (c) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (d) $2-\sqrt{3}$ (e) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

(f) $-2-\sqrt{3}$ (g) $\frac{-\sqrt{3}-1}{2\sqrt{2}}$ (h) $\frac{-\sqrt{3}-1}{2\sqrt{2}}$ (i) $\sqrt{3}$ (j) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

3. (a) $\frac{\sqrt{3}-1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}+1}{\sqrt{2}}$ (c) 0

8. (a) $\frac{56}{65}$ (b) $\frac{63}{65}$ (c) $\frac{56}{33}$

9. (a) $-\frac{1}{5\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) -7

12. (a) $\frac{16}{162}$ (b) $\frac{4}{5}$ (c) $\frac{m}{m+1}$

Exercise 6.1

1. (a) Show to your teacher.
2. (a) (2, 1) (b) (–3, 11) (c) (6, 8) (d) (2, 3) (e) (–1, –4) (f) (–1, –1)
3. Show to your teacher.
4. (a) $\sqrt{5}$, $\tan^{-1}\left(\frac{1}{2}\right)$; $(-2,-1)$ (b) $\sqrt{130}$, $\tan^{-1}\left(\frac{11}{3}\right)$, $(3,-11)$

(c) 10, $\tan^{-1}\left(\frac{4}{3}\right)$, $(-6,-8)$ (d) $\sqrt{13}$, $\tan^{-1}\left(\frac{4}{3}\right)$, $(-2,-3)$

5. (a) $\begin{pmatrix}9\\-1\end{pmatrix}\left(\frac{9}{\sqrt{82}},\frac{-1}{\sqrt{82}}\right)$ (b) $\begin{pmatrix}-1\\-1\end{pmatrix}\left(\frac{-1}{\sqrt{101}},\frac{-10}{\sqrt{101}}\right)$

(c) $\begin{pmatrix}15\\5\end{pmatrix}\left(\frac{3}{\sqrt{10}},\frac{1}{\sqrt{10}}\right)$ (d) $\begin{pmatrix}-7\\4\end{pmatrix}\left(\frac{-7}{\sqrt{65}},\frac{4}{\sqrt{65}}\right)$

6. Show to your teacher.
7. (a) (5, 7) (b) (–4, –5) (c) (2, 2) (d) (0, 1)

Exercise 6 .2

1. (a) $\begin{pmatrix}\frac{9}{11}\end{pmatrix}$ (b) $\begin{pmatrix}\frac{-6}{-8}\end{pmatrix}$ (c) $\begin{pmatrix}\frac{-20}{24}\end{pmatrix}$ (d) $\begin{pmatrix}\frac{25}{30}\end{pmatrix}$ (e) 25

   (f) $2\sqrt{61}$ (g) $10+\sqrt{61}$

2. (a) (6, 11) (b) (–6, –2) (c) (8, –5) (d) (0, 0)

   (e) (–2, 3) (f) (6, 6)

3. (a) $\begin{pmatrix}\frac{13}{2}\end{pmatrix}$ (b) $\begin{pmatrix}\frac{0}{-23}\end{pmatrix}$ (c) $4\sqrt{6}$ (d) (5, –1) (e) (–1, –9)

   (f) $\sqrt{82}$ (g) $\sqrt{26}$

4. Show to your teacher.

5. (a) $-\overrightarrow{4i}$ (b) (3, 4)

(c) $\vec{i}-\overrightarrow{3j}\sqrt{10}, \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right)$ (d) (–6, 7), $\sqrt{85}, \left(\frac{-6}{\sqrt{85}}, \frac{7}{\sqrt{85}}\right)$

6, 7 and 8 show to your teacher.

Exercise 7.1

4 a. (4, 2) b. (-4, -2) c. (-2, 4) d. (2, -4) e. (2, -2) f. (4, 2)

5 a. (-2, 3) b. (1, 4) c. (-5, -6) 6 a.(-5, -3) b. (-1, 0) d. (0, -6)

7 a. (-7, -5) and (-5, 2) b. (-6, -1) and (-4, 2) c. (-6, 0) and (-2, -4)

8 a. ((1, 3) b. (-5 , 7) 9 a. (a = $\frac{7}{3}$, b= 8) b. (p= 2, q= 3)

10 a. x-axis b. y = x

11 a. (1, -2) (0, 2) (2, -3) b. (-2, -1) (2 0) (-3, -2)

12 a. (-1, 3) (0, 5) (2, 1) (3, 4) b. (-2, -9) (0, -8) (2, -9) (0, -10)

Exercise 7.2

4 a.(1, 3) b. (-3 , 1) c. (-1 , -3) d. (1, 3) e. (-3, 1) f. (-1, -3)

5 a. (-6, 4) b. (-4, 2) c. (6, -5) 6 a. (-5, -3) b. (0, 5) c. (-4, 0)

7a. (-3, -5) (-1, -2) b. (-2, 1) (0, -2) c.(-2, 0) (2, 4)

8a. 90 +ve b. (7, 5)

9 a. (a = $-\frac{1}{2}$, b = 9) b. (p = 1 , q = 4)

10 a. -90 (-ve) b. 180 or Half turn

11 a. (3 , -1) (-4, 0) (-2, 2) b. (-2, 3) (2. 1) (-3, 0)

12 a. (1, -3) (2, -5) (4, -1) (5 , -4)   b. (0, 2) (2, 3) (3, -1) (4, -3)

Exercise 7.3

4a. (5, 2)   b. (-1, 7)

11 . (i) (5, -3) (1, -2) (6, 1)   (ii) (4, -4) (7, -3) (6, -6) , (3, -6)

b. (0, -5), (4, -4), (6, -6), (2, -7)

Exercise 7.4

4. a.(-4, -6)   b. $\left(-\frac{2}{3}, \frac{5}{3}\right)$ c. (-9, -5)

5. a. $x = 3$, y = 2   b. p = 3, q = 3

6. a. $\frac{1}{2}$ b. k = 1

7. a (4, 2) (-4, 2) (-6, 2)   b. (0, 0) $(\frac{3}{2}, \frac{3}{2})$ $(0, \frac{3}{2})$

8. a (3, -2) (3, 0) (-3, 6) (-7, 2)   b. (-7, -6) (-1, -12) (-1, -9) (2, -6)

Exercise 8.1

2. (a) 26   (b) 75   3 (a) 20   (b) 20   4. (a) 42   (b) 59

5. (a) 6   (b) 20   6. (a) 40   (b) 45   7. (a) 16   (b) 40

8. (a) 23.75, 32, 52.25   (b) 23, 39, 48

9. (a) 69.75, 60, 60   (b) 24.17, 34, 50.45

10. (a) 5, 11   (b) 12, 25   11.(a) 45, 45, 45   (b) 55, 55, 55

Exercise 8.2 (A)

2. (a) 11.5, 0.333   (b) 19.35, 0.60   (c) 61   (d) 40

(e) 65.90   (f) 8.44

3. (a) 46.5, 77.5   (b) 54.4, 105.6

4. (a) 15.5, 0.216   (b) 2, 0.1404   (c) 3.625, 0.19

5. (a) 4, 0.2   (b) 10, 0.25   (c) 10, 0.222

Exercise 8.2 (B)

1. (a) 12.48, 0.17   (b) 1.5, 0.103   (c) 2.83, 0.145

2. (a) 12.42, 0.177   (b) 1.5, 0.1   (c) 2.76, 0.145

3. (a) 3.66, 0.189   (b) 12.29, 0.34   (c) 55.12, 1.17

4. (a) 24.39, 0.97 (b) 20.46, 6.821 (c) 11.82, 0.295

Exercise 8.2 (C)

3.	(a) 11.33, 0.36	(b) 3.52, 0.15	(c) 8.92, 0.26
4.	(a) 6.81, 0.172	(b) 3.9 , 0.05	(c) 4.78, 0.054
5.	(a) 34.75, 0.25	(b) 33.78, 0.29	(c) 74.64, 23.9
6.	(a) 63.01, 19.70	(b) 217.46, 24.7	
7.	(a) 39.88, 52.22	(b) 37.13, 61.89	
8.	(a) 5.76, 12.37	(b) 10.49, 6.9	